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TABLES OF THE PROBABILITY INTEGRALS OF SYMMETRICAL FREQUENCY CURVES IN THE CASE OF LOW POWERS SUCH AS ARISE IN THE THEORY OF SMALL SAMPLES.

BY KARL PEARSON, ASSISTED BY BRENDA STOESSIGER, M.Sc.

(i) THE symmetrical curves to be considered are those for which $\beta_1 = 0$ and β_2 takes any value from 1 to ∞. The curves are supposed completely determined by β_2 and their standard deviations.

Their differential equation will be

$$\frac{1}{y}\frac{dy}{dx'} = \frac{2mx'}{c_0 + x'^2},$$

leading to

$$y = y_0(c_0 + x'^2)^m,$$

where

$$c_0 = \frac{2\beta_2}{\beta_2 - 3}\sigma^2, \text{ and } m = \frac{1}{2}\frac{9 - 5\beta_2}{\beta_2 - 3}.$$

We can throw them into the following forms:

(i) $\beta_2 = 1$ to $1{\cdot}8$ ($m_1 = 1$ to 0),

$$y = y_0 \frac{1}{\left(1 - \frac{x'^2}{a_1^2}\right)^{m_1}} \quad \text{.....(i)},$$

where

$$a_1^2 = \frac{2\beta_2}{3 - \beta_2}\sigma^2, \text{ and } m_1 = \frac{1}{2}\frac{9 - 5\beta_2}{3 - \beta_2}.$$

This symmetrical curve passes from two equal lumps through U-curves to a rectangle.

(ii) $\beta_2 = 1{\cdot}8$ to 3 ($m_2 = 0$ to ∞),

$$y = y_0\left(1 - \frac{x'^2}{a_2^2}\right)^{m_2} \quad \text{.....(ii)},$$

where

$$a_2^2 = \frac{2\beta_2}{3 - \beta_2}\sigma^2, \text{ and } m_2 = \frac{5\beta_2 - 9}{2(3 - \beta_2)}.$$

This type of curve passes from a rectangle through limited range curves to the normal curve ($\beta_2 = 3$).

(iii) $\beta_2 = 3$ to ∞ ($m_2 = \infty$ to $\frac{5}{2}$),

$$y = y_0 \frac{1}{\left(1 + \frac{x'^2}{a_3^2}\right)^{m_3}} \quad \text{.....(iii)},$$

where

$$a_3^2 = \frac{2\beta_2}{\beta_2 - 3}\sigma^2, \text{ and } m_3 = \frac{1}{2}\frac{5\beta_2 - 9}{\beta_2 - 3}.$$

The limit $\beta \to \infty$ occurs when $m_3 = \frac{5}{2}$ and $a_3^2 = 2\sigma^2$. This curve passes from the normal curve through all grades of leptokurtosis. The limits of range in (i) are from $-a_1$ to $+a_1$, in (ii) from $-a_2$ to $+a_2$, and in (iii) from $-\infty$ to $+\infty$.

We will now proceed to the probability integral of these three curves.

For (i) we have

$$ {}_1P_x = \tfrac{1}{2} + \frac{\int_0^{x'} y\,dx'}{2\int_0^{a_1} y\,dx'} = \tfrac{1}{2}\left\{1 + \frac{B_x(\frac{1}{2}, (1-m_1))}{B(\frac{1}{2}, (1-m_1))}\right\} $$

$$ = \tfrac{1}{2}\{1 + I_x(\tfrac{1}{2}, (1-m_1))\}, $$

where $B_x(p, q)$ is the incomplete and $B(p, q)$ the complete B-function, and $I_x(p, q)$ their ratio.

The required transformation is

$$ x = x'^2/a_1^2, \quad \text{or} \quad = \frac{x'^2}{\sigma^2}\frac{3-\beta_2}{2\beta_2}. $$

Now m_1 lies between 0 and 1, and accordingly to obtain the probability integral of the curve (i) we have only to add unity to the B-function ratio $I_x(\frac{1}{2}, (1-m_1))$ and divide by two.

Since m_1 only lies between 0 and 1, this involves the tabulation of $I_x(\frac{1}{2}, (1-m_1))$ for small ranges of m_1; but this has not yet been completed, and we cannot at present provide a table of the probability integral of the symmetrical curve (i).

Meanwhile, and until the required table be completed, a good method to determine $I_x(\frac{1}{2}, (1-m_1))$ is to use the formula provided by Soper* for the integral

$$ \int_0^{x'} x'^{p-1}(1-x')^{q-1}dx', $$

when p and q are small.

We shall not further consider the probability integral of the curve (i).

For (ii) we have to make the same transformation,

$$ x = \left(\frac{x'}{a_2}\right)^2 = \frac{x'^2}{\sigma^2}\frac{3-\beta_2}{2\beta_2}, $$

and have $$ {}_2P_x = \tfrac{1}{2}\{1 + I_x(\tfrac{1}{2}, (1+m_2))\}. $$

Table I gives the value of

$$ \tfrac{1}{2}\{1 + I_x(\tfrac{1}{2}, \tfrac{1}{2}(n-1))\}, $$

and accordingly we must take $n = 2m_2 + 3$; it runs from $m_2 = -\frac{1}{2}$ to $m_2 = 14$.

When $m_2 = 14$, $\beta_2 = 2{\cdot}818{,}182$, and we have not yet reached closely enough the normal curve ($\beta_2 = 3$) to use its probability integral as anything but a rough approximation.

* *Tracts for Computers*, No. VII. pp. 21—22, Cambridge University Press. See also *Tables for Statisticians and Biometricians*, Part II.

For (iii) the requisite transformation is

$$\frac{x'^2}{a_3^2} = \frac{x}{1-x}, \quad \text{or} \quad x = \frac{x'^2}{x'^2 + a_3^2},$$

and we have $${}_3P_{x'} = \tfrac{1}{2}\{1 + I_x(\tfrac{1}{2}, (m_3 - \tfrac{1}{2}))\};$$

our table will accordingly give ${}_3P_{x'}$ from $m_3 = 15{\cdot}5$ to $m_3 = 2{\cdot}5$, or from $\beta_2 = 3{\cdot}230{,}769$ to $\beta_2 = \infty$. The former value of β_2 is still too far from $\beta_2 = 3$ to allow anything but a rough approximation to be obtained from the normal curve.

If we choose our curve to be $$y = \frac{y_0}{(1 + x''^2)^{n/2}}$$

as is frequently done, then $n = 2m_3$, and

$$a = \sqrt{\frac{2\beta_2}{\beta_2 - 3}}\,\sigma = \sqrt{2m_3 - 3}\,\sigma = \sqrt{n-3}\,\sigma,$$

or $$\sigma = \frac{1}{\sqrt{n-3}} \text{ if we take } a = 1.$$

Accordingly at the end of Table I we have placed the probability integral of the normal curve with a standard deviation $\dfrac{1}{\sqrt{n-3}}$, where $n = 31$, for comparison with that of the curve

$$y = \frac{y_0}{(1 + x''^2)^{15{\cdot}5}}.$$

The result confirms the inference drawn from the value of β_2, i.e. that the normal curve will only give a rough approximation to the exact probability integral at $n = 31$. At the top of the table we may be in error in two to three units in the third place of decimals*.

(ii) We will now describe the two tables here provided.

Table I gives the value of

$$\tfrac{1}{2}\{1 + I_x(\tfrac{1}{2}, \tfrac{1}{2}(n-1))\},$$

where the argument x increases by ·01.

We need to know the relations between the m's and n's, and x and x'.

Curve (i). m_1 lies between 0 and 1, and the only values available in our table are for $n = 2$ and 3, or $m_1 = 0{\cdot}5$ and 0, while x is determined by $x = \dfrac{x'^2}{a_1^2}$.

Curve (ii). m_2 ranges from 0 to ∞, but the table only supplies values from 0 to 14, since $m_2 = \frac{1}{2}(n-3)$. x is found from $x = \dfrac{x'^2}{a_2^2}$.

Curve (iii). m_3 ranges from 2·5 to ∞, or our table will supply the probability integrals of this curve from 2·5 to 15·5. The x is to be found from $x = \dfrac{x'^2}{x'^2 + a_3^2}$.

* Actually the unpublished tables of the B-function will carry us up to $n = 101$, $m_3 = 50{\cdot}5$, a value which gives a much closer approximation to a normal curve.

When the curve is written in the form

$$y = \frac{y_0}{(1+z^2)^{\frac{1}{2}n}},$$

our table will supply the probability integrals for $n = 5$ to 31. If we choose to neglect the infinity of the fourth moment we can proceed to $n = 2$.

In the last form of this curve $x = \dfrac{z^2}{1+z^2}$, or $z^2 = x/(1-x)$. This value of z^2 is given to five decimal places in the second column of each sheet of the table. This enables the user to ascertain rapidly whereabouts he is in the x-variate for a given value of z or z^2.

(iii) We need two kinds of interpolation into Table I: (*a*) we need to interpolate between the tabulated values of n, and (*b*) we need to interpolate between the tabulated values of x. Both these interpolations give rise to difficulties, which require some consideration.

(*a*) After $n = 8$, interpolations for n lying between tabled values are successful, if we use δ^2 and occasionally δ^4. Neither Table I (nor the supplementary Table II) will give satisfactory brief interpolations for n less than 8. It may even be doubted, if the argument n were tabled by 0·1 instead of 1·0, whether satisfactory brief interpolation could be achieved. Although the graphs of the function for constant x give very simple smooth curves, after many trials no short interpolation process has been yet discovered. Luckily the chief use of the present tables is their application to small samples, and in such cases n is a whole number. For interpolation by the forward difference formulae, see the Appendix to this paper.

(*b*) With regard to direct interpolation for x, this is feasible for $x = \cdot 11$ onward throughout the table using δ^2, or occasionally if greater accuracy be required δ^2 and δ^4. But from $x = \cdot 00$ to $\cdot 10$, ordinary interpolation formulae cannot be applied, owing to the infinite differential coefficients appearing with the factor $x^{-\frac{1}{2}}$ in the integral. Accordingly an auxiliary table—Table II—has been formed which gives the function

$$\mathscr{P}_x(n) = \frac{P_x(n) - 0{\cdot}5}{\sqrt{x}},$$

and provides its δ^2*. This will suffice to ascertain $\mathscr{P}_x(n)$ for any value of x from $\cdot 00$ to $\cdot 10$, and therefore

$$P_x(n) = \mathscr{P}_x(n)\sqrt{x} + 0{\cdot}5.$$

The user of Table II must therefore find the square root of the argument† with which he enters it, as the multiplier for $\mathscr{P}_x(n)$.

* Determined from the nine-figure B-function Table. For $\delta^2\mathscr{P}_0(n)$ we used the formula

$$\delta_0^2 = 4\delta_1^2 - 6\delta_2^2 + 4\delta_3^2 - \delta_4^2.$$

† x will not generally exceed four decimals, so that any table of square roots will provide what is required.

(iv) Illustrations of the use of the Tables.

Illustration (i). The frequency curve for the distribution of the correlation coefficient r in samples of size p taken from a parent population in which the correlation is zero is given by the curve

$$y = y_0 (1 - r^2)^{\frac{1}{2}(p-4)},$$

where the mean, $\bar{r}$, $= 0$, and since $\alpha_2 = 1$, $\sigma = \dfrac{1}{\sqrt{p-1}}$. What is the chance that in a sample of 20,

(a) r will lie outside twice its standard deviation?

(b) r will lie outside the limits $\pm$ ·50?

The above curve is our Type (ii), and therefore $m_2 = \frac{1}{2}(p-4) = 8$ for this special case. Now $m_2 = \frac{1}{2}(n-3) = 8$, and accordingly $n = 19$. The proper transformation is $r^2 = x$. We have $\sigma = \dfrac{1}{\sqrt{19}} = ·229,4157$.

If $r = 2\sigma = ·458,8314$, then $x = r^2 = ·210526$. If $r = ·50$, $x = ·25$.

We have accordingly to find from Table I, for $n = 19$, the value of the function tabled for $x = ·210526$ and $x = ·25$.

The latter comes without interpolation at once as $\frac{1}{2}(1 + \alpha_2) = ·987,6152$, or $\frac{1}{2}\alpha_2 = ·487,6152$, hence doubling, we find the chance is ·975,2304, or the odds are about 975 to 25, or 39 to 1, that in taking a sample of 20 individuals from a normal population two characters of zero correlation will not show a correlation in the sample exceeding numerically $\pm$ ·50.

In the first we have to interpolate between the values for x of ·21 and ·22, i.e.

$$u_0 = ·978,9245, \qquad u_1 = ·981,5217,$$
$$\delta^2 u_0 = -3461, \qquad \delta^2 u_1 = -3059.$$

Fourth differences are here unnecessary.

$$\theta = ·0526, \quad \phi = ·9474, \quad \tfrac{1}{6}\theta\phi = ·0083,0554,$$
$$u_\theta = ·9790,6111 + 0000,0827 = ·979,0694.$$

The chance therefore of r falling within the range $\pm 2\sigma$ is ·958,1388. Had we assumed the distribution of r to be a normal curve, the chance of r falling within the range $\pm 2\sigma$ would be ·954,4998.

Illustration (ii). In a sample of 12, the correlation coefficient is found to be ·3. What is the chance that in the original population there was no correlation?

In this case $\qquad m_2 = \frac{1}{2}(p-4) = 4 = \frac{1}{2}(n-3),$

and $\qquad n = 11, \quad x = r^2 = ·09.$

Our table under $n = 11$ gives for $x = ·09$ the value ·828,2807. The chance accordingly, of r exceeding $\pm$ ·30, if the correlation were zero, would be

$$2\,(1 - ·828,2807),$$

or if the population sampled had no correlation between the variants considered, a correlation of numerical intensity ·30 or more would occur in 343 out of 1000

samples, i.e. in more than one sample in three. We cannot therefore assert that the correlation found in the sample marks a significant correlation in the parent population.

Even if the observed correlation in the sample were ·50, there would still be 98 samples in 1000 with a correlation of $\pm$ ·50 or more if the parent population had no correlation. Indeed correlation coefficients found from very small samples are of small service in indicating significant correlation in the parent population unless the correlation in the sample be very high. For example, if the correlation in the sample of 12 were ·80, samples from an uncorrelated population would only give rise to such a value once in 500 trials.

Illustration (iii). What is the chance in a sample of 31 that the regression coefficient will not differ from that of the parent population, supposed normal, by more than twice its standard deviation?

If ρ be the correlation, Σ_1, Σ_2 the standard deviations in the parent population and R_1 the regression coefficient in the sample, the distribution of R_1 is given by

$$y = \frac{y_0}{\left\{(n-3)\,\sigma_{R_1}{}^2 + \left(R_1 - \rho\frac{\Sigma_1}{\Sigma_2}\right)^2\right\}^{\frac{1}{2}n}}$$

$$= \frac{y_0'}{\left\{1 + \frac{x'^2}{n-3}\right\}^{\frac{1}{2}n}},$$

where $\bar{R}_1 = \text{Mean } R_1 = \rho\frac{\Sigma_1}{\Sigma_2}$, the value of the regression in the parent population,

$$\sigma_{R_1}{}^2 = \frac{1}{n-3}\frac{\Sigma_1{}^2}{\Sigma_2{}^2}(1-\rho^2),$$

and

$$x' = \frac{\text{Deviation of } R_1 \text{ from } \bar{R}_1}{\text{Standard Deviation of } R_1},$$

n being the size of the sample.

The requisite transformation is

$$x'^2/(n-3) = x/(1-x) \quad \text{or} \quad \frac{x'^2}{x'^2 + a_3{}^2} = x.$$

Thus if $x' = 2$, we have

$$x = \frac{4}{4+n-3} = \frac{4}{n+1} = \text{in our case } \tfrac{1}{8} = \cdot 125.$$

We have accordingly to compute

$${}_3P_{x'} = \tfrac{1}{2}\{1 + I_{\cdot 125}(\tfrac{1}{2}, \tfrac{1}{2}(n-1))\}.$$

The value will be found in the column for $n = 31$, or $\frac{1}{2}(n-1) = 15$, between the values of ·12 and ·13 of x. We have

$$u_0 = \cdot 973{,}9461, \quad \delta^2 u_0 = -\,10529, \quad \delta^4 u_0 = -\,517,$$

$$u_1 = \cdot 978{,}6801, \quad \delta^2 u_1 = -\,\;8559, \quad \delta^4 u_1 = -\,396.$$

We are therefore at a part of the table where it is requisite to use δ^4's as well as δ^2's, if we desire an accurate value of $P_{x'}$. Now

$$\theta = \cdot 5, \quad \phi = \cdot 5, \quad \tfrac{1}{6}\theta\phi = \cdot 041{,}6667,$$

and
$$\begin{aligned} u_\theta &= \tfrac{1}{2}(\cdot 973{,}9461 + \cdot 978{,}6801) + \cdot 041{,}6667 \times 1\cdot 5\,(\cdot 001{,}9088) \\ &\quad - \cdot 041{,}6667 \times \cdot 1125 \times 2\cdot 5\,(\cdot 000{,}0913) \\ &= \cdot 976{,}3131 + \cdot 000{,}1193 - \cdot 000{,}0011 \\ &= \cdot 976{,}4313. \end{aligned}$$

Hence $\cdot 952{,}8626$ is the chance that the regression coefficient will lie within $\pm$ twice its standard deviation from the true value in the parent population.

Illustration (iv). In a long series of observations on Fathers and Sons the correlation coefficient for span was found to be $\cdot 454$, and the standard deviations were $3''\cdot 14$ and $3''\cdot 11$ respectively. The regression R_1 of Son on Father for span $= \cdot 44976$. The standard deviation of R_1 in samples is

$$\sigma_{R_1}{}^2 = \frac{1}{n-3}\,\frac{(3''\cdot 11)^2}{(3''\cdot 14)^2}\,(1 - (\cdot 4497)^2),$$

or
$$\sigma_{R_1} = \frac{1}{\sqrt{n-3}} \times \cdot 884{,}666.$$

Hence, if we take this to be a reasonable normal parent population for span, let us ask whether a sample of 19 pairs of Father and Son giving a correlation of $\cdot 390$ and standard deviations for span: Fathers $3''\cdot 19$ and Sons $2''\cdot 98$, may be supposed reasonably to have been drawn from this parent population.

Now R_1 for the sample $= \cdot 36432$ and $\sigma_{R_1} = \cdot 221{,}1665$.

Thus
$$x' = \frac{\cdot 36432 - \cdot 44966}{\cdot 221{,}1665} = -\cdot 385{,}863.$$

Accordingly
$$x'^2 = \cdot 1488{,}9025,$$

and
$$x = \frac{\cdot 1488{,}9025}{\cdot 1488{,}9025 + 16} = \cdot 0921{,}9844.$$

This clearly lies within the first part of Table I where the differences are unsatisfactory. We therefore use the auxiliary Table II. For $n = 19$, we have

$$u_0 = 1\cdot 335{,}4038, \quad \delta^2 u_0 = 12631,$$
$$u_1 = 1\cdot 305{,}4459, \quad \delta^2 u_1 = 12083.$$

δ^4's will be unnecessary.

$$\theta = \cdot 219{,}844, \quad \phi = \cdot 780{,}156, \quad \tfrac{1}{6}\theta\phi = \cdot 028{,}5854,$$
$$\begin{aligned} u_\theta &= 1\cdot 328{,}8177 - \cdot 000{,}1064 \\ &= 1\cdot 328{,}7113 = \mathscr{P}_x(19). \end{aligned}$$

But
$$P_x(19) = \cdot 5 + \mathscr{P}_x(19)\sqrt{x},$$
$$P_x(19) = \cdot 5 + \cdot 303{,}6420 \times 1\cdot 328{,}7113 = \cdot 903{,}452,$$

or the chance if the above sample were really drawn from the above parent population that its regression coefficient would differ as much as or more than it does do from the regression in the parent population $= \cdot 193{,}096$.

We see therefore that in about 19 in 100 samples the deviation of the regression would be greater than that observed.

Let us, however, look at this problem in another way, which will illustrate a further application of our present table.

Illustration (v). In the sample of the previous illustration the first product moment coefficient $= p_{11} = \cdot390 \times 2\cdot98 \times 3\cdot19 = 3\cdot707,4180$. What is the chance that a sample of 19 with this p_{11} could have been extracted at random from a parent population with no correlation, but with standard deviations 3″·14 and 3″·11?

We compute v: $$v = n\frac{p_{11}}{3\cdot14 \times 3\cdot11} = \frac{19\,(3\cdot707,4180)}{9\cdot7654}$$
$$= 7\cdot213,3187,$$

then the problem reduces to determining the chance that values of v will differ from zero by an amount as great as or greater than this. The distribution of v is given by

$$y_v = \frac{N}{\sqrt{\pi}\,(n^2-1)}\,\frac{\Gamma\left(\frac{1}{2}(n+4)\right)}{\Gamma\left(\frac{1}{2}(n+3)\right)}\,\frac{1}{\left(1+\frac{v^2}{n^2-1}\right)^{\frac{1}{2}(n+4)}},$$

where $$v = n\frac{p_{11}}{\sigma_1\sigma_2},$$

and σ_1, σ_2 are the standard deviations in the parent population. The curve obviously falls under our Type (iii) above.

We write $$y = y_0\frac{1}{\left(1+\frac{v^2}{n^2-1}\right)^{\frac{1}{2}(n+4)}}$$
$$= \tilde{\sigma}_0\left(1+\frac{v^2}{360}\right)^{-11\cdot5}.$$

We have accordingly to take $m_3 = 11\cdot5$, and $a_3^2 = 360$, which gives*

$$n = 23,$$
$$x = \frac{52\cdot031,967}{360+52\cdot031,967} = \cdot1262,8138.$$

Hence from column for $n = 23$ of Table I we find

$$u_0 = \cdot951,3679, \quad \delta^2 u_0 = -\,11583, \quad \delta^4 u_0 = -\,407,$$
$$u_1 = \cdot958,2584, \quad \delta^2 u_1 = -\,\;9804, \quad \delta^4 u_1 = -\,310,$$
$$\theta = \cdot628,138, \qquad \phi = \cdot371,862, \quad \tfrac{1}{8}\theta\phi = \cdot038,9301,$$
$$u_\theta = \cdot955,6961 + \cdot000,1240 - \cdot000,0008$$
$$= \cdot955,8193.$$

Thus in 884 out of 10,000 samples a v and therefore a p_{11} *numerically* as large or larger than the observed product moment coefficient could have arisen from a parent population without correlation. The odds are therefore only about 116 to

* This n is that of the Tables, and not the n above which is the size of the sample, the former n = the latter $n+4$.

10 that p_{11} did not arise from a population without correlation. It would occur about once in 11 trials. We cannot assert significance in the observed

$$p_{11} = 3{\cdot}707{,}4180.$$

It is well to investigate the significance of the correlation observed.

The correlation is ·390 and the size of the sample 19. The distribution curve will then be

$$y = y_0 (1 - r^2)^{7{\cdot}5},$$

and

$$m_2 = 7{\cdot}5 = \tfrac{1}{2}(n-3), \text{ or } n = 18,$$

$$x = r^2 = {\cdot}1521.$$

Turning to our Table I:

$$u_0 = {\cdot}949{,}3160, \quad \delta^2 u_0 = -7531,$$

$$u_1 = {\cdot}955{,}1406, \quad \delta^2 u_1 = -6616,$$

$$\theta = {\cdot}21, \quad \phi = {\cdot}79, \quad \tfrac{1}{6}\theta\phi = {\cdot}02765,$$

and the use of δ^4 is unnecessary. Accordingly

$$u_\theta = {\cdot}950{,}5392 + {\cdot}000{,}0594$$
$$= {\cdot}950{,}5986.$$

The chance is therefore $1 - 2\,({\cdot}950{,}5986 - {\cdot}5) = {\cdot}098{,}8028$ that a sample of 19 from a population of zero correlation would show a correlation *numerically* greater than ·390. Thus such a correlation will occur in samples of this size about once in 10 trials.

It will be clear from the results in this illustration:

(*a*) That the introduction of the observed standard deviations into the sample (i.e. using $p_{11} = r\sigma_1\sigma_2$ instead of r) lessens the probability of the parent population being one of zero correlation.

(*b*) That very little of definite value can be learnt as to correlation from small samples, i.e. in the above illustrations the sample might have been easily obtained from a parent population of correlation = ·00 or ·45 *.

Illustration (vi). In the long series of observations referred to in Illustration (iv) the mean spans of Fathers and of Sons were 68″·67 and 69″·94 respectively. Hence the regression line of Son's span on Father's span is

$$\tilde{y} = 39''{\cdot}06 + 0''{\cdot}44966x.$$

If $\tilde{y}_x$ be the value of $\tilde{y}$ found in a particular sample from the regression line of that sample, the standard deviation of $\tilde{y}_x$'s for numerous samples is

$$\sigma^2_{\tilde{y}_x} = \frac{\Sigma_2^2(1-\rho^2)}{n-3}\left\{1 - \frac{2}{n} + \left(\frac{x-m_1}{\Sigma_1}\right)^2\right\}$$

$$= \frac{(3{\cdot}11)^2(1-({\cdot}454)^2)}{n-3}\left\{1 - \frac{2}{n} + \frac{(x-m_1)^2}{(3{\cdot}14)^2}\right\}$$

$$= \frac{7{\cdot}678{,}5254}{n-3}\left\{1 - \frac{2}{n} + \frac{(x-68''{\cdot}67)^2}{9{\cdot}8596}\right\}.$$

* Inferences like these in character may easily be drawn by looking at Table I for $n = 19$ and examining the entries above + ·39 and below − ·39 in the column with $\rho = 0$, ·4 and ·5.

Now suppose we fix our attention on Fathers with spans between 66″ and 67″, i.e. put $x = 66''{\cdot}5$, and suppose samples taken of size 19. Then

$$\sigma^2_{\tilde{y}_x} = {\cdot}4799{,}0784\,\{1 - {\cdot}1052{,}6316 + {\cdot}4775{,}9544\}$$
$$= {\cdot}6585{,}9302,$$

and $$\sigma_{\tilde{y}_x} = {\cdot}811{,}5374.$$

For $x = 66''{\cdot}5$, we have $\tilde{y} = 68''{\cdot}96$

from the regression line.

Now we will suppose the regression line for the sample of 19 (!) has been found and gives for the mean span of Sons of Fathers of 66″ to 67″ span the value $\tilde{y}_x = 68''{\cdot}26$. The parent population gives 68″·96. Is this a reasonable difference?

The distribution of $\tilde{y}_x - \tilde{y}$ will be given by the curve

$$y = \frac{y_0}{\left\{1 + \frac{1}{n-3}\left(\frac{\tilde{y}_x - \tilde{y}}{\sigma_{\tilde{y}_x}}\right)^2\right\}^{\frac{1}{2}n}},$$

and we have $$x' = \frac{\tilde{y}_x - \tilde{y}}{\sigma_{\tilde{y}_x}} = \frac{{\cdot}70}{{\cdot}811{,}5374},$$

or $$x'^2 = {\cdot}74401.$$

Thus $$x = \frac{x'^2}{x'^2 + a_3{}^2} = \frac{{\cdot}74401}{16{\cdot}74401} = {\cdot}04443.$$

We have accordingly to interpolate from our tables for $x = {\cdot}04443$ in the column $n = 19$. This for accuracy must be done by aid of Table II.

We have $u_0 = 1{\cdot}505{,}3176$, $\delta^2 u_0 = 15731$, $\delta^4 u_0 = 27$,

$u_1 = 1{\cdot}468{,}4491$, $\delta^2 u_1 = 15060$, $\delta^4 u_1 = 27$.

Clearly we need not use δ^4's.

$$\theta = {\cdot}443, \quad \phi = {\cdot}557, \quad \tfrac{1}{6}\theta\phi = {\cdot}041{,}1252,$$
$$u_\theta = 1{\cdot}488{,}98485^+ - {\cdot}000{,}19010 = 1{\cdot}488{,}7948.$$

Thus $$\mathscr{P}_x = 1{\cdot}488{,}7948,$$

and $$P_x = {\cdot}5 + \sqrt{{\cdot}04443} \times 1{\cdot}488{,}7948$$
$$= {\cdot}813{,}8145.$$

Hence assuming the sample to lie within the range $\pm\, 0''{\cdot}7$ from the value 68″·96 for Sons of Fathers having spans of 66″ to 67″ in the sampled population, the chance of a deviation *numerically* as large as or larger than this $= 2\,(1 - P_x) = {\cdot}372{,}3710$, or we might expect 37·2 % of samples of 19 to give a worse disagreement with the value in the sampled population.

N.B. The reader will note that we are *not* comparing the mean of actual isolated individuals in the sample with Fathers having spans between 66″ and 67″, but we are comparing the mean of the Sons of this array of Fathers *found from the regression line of the sample* with the value of the same mean as given by the parent population.

We can use our tables as applied to the third type of curve to test whether a sample of which we know the mean and standard deviation comes from a parent population of which we know the mean.

Let the size of the sample be n, the mean and standard deviation of the sample be m and s, and the mean of the parent population be M. Then, if

$$x' = (m - M)/s,$$

the distribution of x' in samples of size n is given by*

$$y = \frac{y_0}{(1 + x'^2)^{\frac{1}{2}n}},$$

provided the parent population be normally distributed. E. S. Pearson has shown the extent to which this result may still be applied in a certain range of non-normal distributions†.

It is difficult to imagine a practical case in which we know M so accurately that its probable error relative to that of m is negligible, and yet do not know Σ the standard deviation of the parent population with corresponding accuracy. If we know both M and Σ we have two *independent* variables m and s to compare with them, and the writer of the present paper personally much prefers in all such cases the double test to the single test which involves both characters.

Illustration (vii). Among samples of 10 from a normal population of mean variate zero and standard deviation 10, a sample occurred with mean 7·0 and standard deviation 14·64‡. What is the probability of such a sample occurring *at a single draw* as judged by the present test?

$$x' = \frac{7{\cdot}0}{14{\cdot}64} = {\cdot}4781, \text{ and } x'^2 = {\cdot}2286.$$

The distribution curve of x' is

$$y = y_0 \frac{1}{(1 + x'^2)^{\frac{1}{2}n}},$$

and the proper transformation $x = \dfrac{x'^2}{1 + x'^2} = {\cdot}1861.$

Turning to Table I under $n = 10$ and $x = {\cdot}1861$, we have

$$u_0 = {\cdot}903{,}2890, \quad \delta^2 u_0 = -4832,$$

$$u_1 = {\cdot}909{,}9040, \quad \delta^2 u_1 = -4443,$$

$$\theta = {\cdot}61, \quad \phi = {\cdot}39, \quad \tfrac{1}{6}\theta\phi = {\cdot}03965,$$

$$u_\theta = {\cdot}907{,}3241{,}5 + {\cdot}000{,}0549{,}9$$

$$= {\cdot}907{,}3791.$$

* This is the case really proved by "Student," *Biometrika*, Vol. v. pp. 7—8; however, the actual examples he gives do not belong to this case, but indicate that he foresaw a wider application of it.

† *Biometrika*, Vol. XXI. pp. 259 *et seq.*

‡ Such a sample was one of a set of 700 samples actually drawn from a normal population.

Thus the chance that a value of x' should occur as large as or larger than this is ·185,2418, taking positive and negative excesses in x' together. The odds are only about 4·5 to 1 against such occurrence.

Now let us consider the two characters m and s which have been combined in "Student's" test separately.

The means in the samples are distributed normally with standard deviation of $\Sigma/\sqrt{n}$ = in our case 3·1623, or the ratio of the observed deviation in the sample mean to the standard deviation of sample means is 2·2136.

From Table II of Part I of these *Tables for Statisticians*:

$$u_0 = \cdot 986{,}4474, \quad \delta^2 u_0 = -77,$$

$$u_1 = \cdot 986{,}7906, \quad \delta^2 u_1 = -75,$$

$$\theta = \cdot 36, \quad \phi = \cdot 64, \quad \tfrac{1}{0}\theta\phi = \cdot 0384.$$

Accordingly
$$u_\theta = \cdot 986{,}5709{,}5 + \cdot 000{,}0008{,}8$$
$$= \cdot 986{,}5718.$$

Thus the chance of a mean as great as or greater than this occurring = ·013,4282, or taking both positive and negative excesses = ·026,8564. Thus the odds against such a mean occurring in a single sample are of the order 36 to 1, while those as judged by "Student's" test are about 4·5 to 1.

Now turn to the standard deviation, which is 14·64 against the 10 of the parent population.

If we judged roughly, assuming the distribution of standard deviations to be approximately normal with a standard deviation $\dfrac{\Sigma}{\sqrt{2n}} = 2{\cdot}2361$ about a mean of $\Sigma = 10$, the deviation $14{\cdot}64 - 10 = 4{\cdot}64$ would be 2·075 times the standard deviations, or deviations as great as or greater than this would only occur about 38 times in 1000 trials, or the odds are of the order 25 to 1 against such an occurrence.

For a more accurate appreciation of the odds, we must note that the curve of distribution of s in samples from a normal population is

$$y = y_0 x'^{n-2} e^{-\frac{1}{2}x'^2},$$

where $x' = s/(\Sigma/\sqrt{2n})$ in our present notation. But this curve has not yet had its probability integral tabled for various values of n and x'.

If, however, we write $z = \frac{1}{4}x'^2$, the probability integral becomes

$$P(z, n) = \frac{\int_0^z z^{\frac{n-3}{2}} e^{-z} dz}{\int_0^\infty z^{\frac{n-3}{2}} e^{-z} dz}$$

= Probability Integral of a Type III curve as tabled in the *Tables of the Incomplete* Γ-*Function**.

* Published by H.M. Stationery Office, 1922.

The integral there given is

$$I(u, p) = \frac{\int_0^{u\sqrt{p+1}} z^p e^{-z} dz}{\int_0^{\infty} z^p e^{-z} dz}.$$

In our case $p = \frac{1}{2}(n-3) = 3{\cdot}5,$

$$u = \frac{1}{\sqrt{4{\cdot}5}}\, z = \frac{1}{4\sqrt{4{\cdot}5}} \left(\frac{s}{\Sigma/\sqrt{2n}}\right)^2 = 5{\cdot}0516.$$

For our interpolation in excess of mean we have from the above tables, under $p = 3{\cdot}5$:

Argument	Entry	δ^2	δ^4
5·0	·988,2633	− 2497	Negligible
5·1	·989,8982	− 2185	,,

$$\theta = {\cdot}516, \quad \phi = {\cdot}484, \quad \tfrac{1}{6}\theta\phi = {\cdot}041{,}624.$$

$$\text{Required value} = {\cdot}989{,}1069{,}1 + {\cdot}000{,}0292{,}1$$

$$= {\cdot}989{,}1361,$$

or the chance of values of s as great as or greater than $14{\cdot}64 = {\cdot}010{,}8639$.

If on the side of defect we take as our limit $14{\cdot}64 - 10 = 4{\cdot}64$, we find $u = {\cdot}5074$. Our tables give us:

Argument	Entry	δ^2	δ^4
0·50	·010,5995	+ 37648	− 2442
0·60	·020,3677	+ 43857	− 2855

$$\theta = {\cdot}074, \quad \phi = {\cdot}926, \quad \tfrac{1}{6}\theta\phi = {\cdot}011{,}4207, \quad \tfrac{1}{240}(1+\theta)(1+\phi) = {\cdot}103{,}4202.$$

$$\text{Required value} = {\cdot}011{,}3223{,}5 - {\cdot}000{,}1366{,}1 - {\cdot}000{,}0015{,}4$$

$$= {\cdot}011{,}1842.$$

Accordingly the probability that s will differ from the population value by as much as or more than 4·64

$$= {\cdot}010{,}8639 + {\cdot}011{,}1842$$

$$= {\cdot}022{,}0481,$$

or the odds are about 44 to 1 against such a deviation from the population standard deviation occurring. Now it would appear that these two sets of odds—36 to 1 against such an excess in the mean and 44 to 1 against such an excess in the standard deviation—especially when we remember that by our hypothesis as to the parent population these two results are independent—are entirely screened when we apply "Student's" test, with its odds of only 4·5 to 1. The fact is that when the two characters, on the ratio of which "Student's" test is based, deviate in the same direction, this test may be very misleading, when we use it as an indication of the rarity of a particular sample; it is the measurement

of the rarity of a particular ratio connected with the sample, but may be dangerous if interpreted as a measure of the rarity of the sample itself*.

That "Student" himself has not laid too great emphasis on his test is, I think, clear, but the emphasis used by others must lead us to be cautious in its application.

While "Student's" analysis follows the lines indicated above of the probability of his ratio in the case of a sample drawn from a normal parent population, he uses it in the examples he gives for a somewhat different purpose, where its application needs some consideration.

Let u and v be two variates, each of which follows the normal law, then their difference $u-v$ will also follow a normal curve with mean $\bar{u}-\bar{v}$ and standard deviation $\sqrt{\sigma_1^2+\sigma_2^2-2r\sigma_1\sigma_2}$, which latter is the standard deviation of the difference, σ_{u-v}, if r be the correlation coefficient of u and v.

Accordingly if we take samples from these populations with means m_u, m_v and standard deviations s_u, s_v and correlation r, then

$$m_u - m_v \text{ and } s_{u-v} = \sqrt{s_u^2 + s_v^2 - 2rs_u s_v}$$

will follow the two curves used by "Student" to obtain his ratio distribution, and if we write

$$x' = \frac{m_u - m_v - (\bar{u} - \bar{v})}{s_{u-v}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\epsilon),$$

then x' will follow the law of distribution in samples of n given by

$$y = \frac{y_0}{(1+x'^2)^{\frac{1}{2}n^*}}$$

"Student" tacitly takes $\bar{u}=\bar{v}$, or he assumes the mean difference of the population from which he is sampling to be zero. He is therefore measuring the probability of the ratio x' on the assumption that u and v are taken at random† from the same parent population. If the ratio x' gives a very small chance of occurrence, he very properly assumes that on his hypothesis u and v are not drawn from the same parent population.

But with "Student" u and v are not *independent* samples of necessity as in J. Neyman and E. S. Pearson's test for two samples (see below).

Take the following series of values from "Student's" original paper:

* A cephalic index among Englishmen of 80·0 is not uncommon, but if we say it has arisen from a skull length of 210 mm. and a skull breadth of 168 mm. we recognise that we are dealing with a very exceptional case on two counts. That is the non-rarity of a ratio is not sufficient to justify us in considering the individual whom it characterises as of common occurrence.

† Actually, however, this is not so, in for example his Illustration I; his two populations are linked by a high correlation due to individual reaction to soporifics. If he gets a high u, he will get a high v and if he gets a low u he will have a low v.

Additional Hours of Sleep gained by the use of hyoscyamine hydrobromide.

Patient	1 (Dextro-)	2 (Laevo-)	Difference (2 – 1)
1	+0·7	+1·9	+1·2
2	−1·6	+0·8	+2·4
3	−0·2	+1·1	+1·3
4	−1·2	+0·1	+1·3
5	−0·1	−0·1	0·0
6	+3·4	+4·4	+1·0
7	+3·7	+5·5	+1·8
8	+0·8	+1·6	+0·8
9	+0·0	+4·6	+4·6
10	+2·0	+3·4	+1·4
Mean	+0·75	+2·33	+1·58
S.D.	1·70	1·90	1·17

Now it is clear that $s_{u-v} = 1{\cdot}17$ is much less than $\sqrt{(1{\cdot}70)^2 + (1{\cdot}90)^2}$, which it should be, were u and v independent. Actually worked out on these ten cases the correlation is over ·79. Is it likely even on ten cases that the correlation would exceed numerically ·79, if it were really zero in the parent population?

The curve of distribution is (see p. 257)

$$y = y_0 (1 - r^2)^3.$$

We have therefore to enter our table with $x = r^2 = {\cdot}6241$, and as $\frac{1}{2}(n-3) = 3$ with $n = 9$, we find that the chance of such a correlation coefficient from a population of zero correlation lying outside the limits $\pm$ ·79 is between ·006 and ·007. There is small doubt therefore that u and v in the sampled population are correlated, probably highly correlated, as the influence of any sleeping draught whatever is a characteristic of the individual. "Student," in applying his test to the difference, has noted this fact as accounting for the low value of the probable error of the difference.

But what, I think, it is desirable to emphasise is that this correlation may exist in most of the examples to which "Student" applies his test, either owing to the influence of the same individual, or of the same year, etc. Accordingly the denominator of "Student's" ratio will be subject to large variation owing to the presence of this correlation in s_{u-v}, the correlation itself being subject to large variation in small samples of such sizes as 10, the numerator $m_u - m_v$ being not subject to this influence of the correlation to the same extent.

Now "Student" takes $x' = 1{\cdot}58/1{\cdot}17 = 1{\cdot}35$, $x'^2 = 1{\cdot}8225$, and accordingly the transformed $x = x'^2/(1 + x'^2) = {\cdot}6457$, while $n = 10$.

Entering our Table I with $n = 10$, we have:

x		δ^2	δ^4
·64	·998,4448	−215	Negligible
·65	·998,6380	−196	"

$$\theta = {\cdot}57, \quad \phi = {\cdot}43, \quad \tfrac{1}{6}\theta\phi = {\cdot}04085.$$

$$\text{Required value} = {\cdot}998{,}5549 + {\cdot}000{,}0025$$
$$= {\cdot}998{,}5574.$$

"Student" gives the value ·9985, quite in keeping.

The chance therefore that x' will not lie between the limits

$$\pm 1{\cdot}35 = 2 \times {\cdot}00144 - {\cdot}0029,$$

or the odds are 9971 to 29 against it or 344 to 1 against it.

Now let us suppose the 10 patients who had dextro-hyoscyamine hydrobromide were not identical with those who had the laevo- form. Then there is no doubt about the application of formula (ϵ). If we suppose them to be independent samples of the same population $r = 0$, and $\bar{u} = \bar{v}$. In this case $m_u - m_v = 2{\cdot}33 - 0{\cdot}75 = 1{\cdot}58$, and

$$s_{u-v} = \sqrt{s_u{}^2 + s_v{}^2} = \sqrt{(1{\cdot}70)^2 + (1{\cdot}90)^2}$$
$$= 2{\cdot}5495.$$

Thus $$x' = 1{\cdot}58/2{\cdot}5495 = {\cdot}6197, \quad x'^2 = {\cdot}3840,$$

and $$x = \frac{x'^2}{1 + x'^2} = {\cdot}2775, \text{ and } n = 10.$$

We have from Table I:

x		δ^2	δ^4
27	·949,3108	−2475	Negligible
28	·952,9130	−2318	,,

$$\theta = {\cdot}75, \quad \phi = {\cdot}25, \quad \tfrac{1}{6}\theta\phi = {\cdot}0336.$$

$$\text{Required value} = {\cdot}952{,}0124 + {\cdot}000{,}0224$$
$$= {\cdot}952{,}0348,$$

or the odds are about 9·4 to 1 that x' does not lie in the range $\pm$ ·6197.

A further test has been provided by J. Neyman and E. S. Pearson* to determine whether two samples, of which the means are m_1, m_2 and the standard deviations s_1 and s_2, have been drawn from the same normal population.

They take $$x' = \frac{m_u - m_v}{\sqrt{n_1 s_1{}^2 + n_2 s_2{}^2}} \sqrt{\frac{n_1 n_2}{n_1 + n_2}},$$

and find its distribution curve to be

$$y = \frac{y_0}{(1 + x'^2)^{\frac{1}{2}(n_1 + n_2 - 1)}}.$$

In the above case of "Student" $n_1 = n_2 = 10$, and $s_1 = 1{\cdot}70$, $s_2 = 1{\cdot}90$, $m_1 = 0{\cdot}75$, $m_2 = 2{\cdot}33$.

Accordingly $$x' = \frac{1{\cdot}58}{2{\cdot}5495\sqrt{2}} = {\cdot}4382, \text{ and } x'^2 = {\cdot}1920,$$

while $$x = \frac{x'^2}{1 + x'^2} = {\cdot}1611,$$

* *Biometrika*, Vol. xxA. pp. 175 *et seq.*

and we must look up the column for $n = 19$ in Table I. We have:

x		δ^2	
·16	·959,7231	− 6554	δ^4 may be neglected
·17	·964,5820	− 5745	for present purposes.

$$\theta = \cdot 11, \quad \phi = \cdot 89, \quad \tfrac{1}{6}\theta\phi = \cdot 016{,}3167.$$

$$\text{Required result} = \cdot 960{,}2576 + \cdot 000{,}0306$$

$$= \cdot 960{,}2882.$$

The chance accordingly of x' exceeding the limits $\pm \cdot 1611$ is ·0794, or the odds against this are about 11·6 to 1.

This is roughly in keeping with the previous determination. Or, we conclude that there is some, but far from overwhelming, evidence that a population treated with the laevo- form of the soporific would have longer hours of sleep than another sample of the same population treated with the dextro- form. On the other hand, if we can trust the application of formula (ϵ) to the case where the samples are not independent samples, then the odds are 344 to 1 that the *same* individual gets longer hours of sleep from the laevo- than from the dextro- form.

The difference lies and can lie only in the correlation in the individual between hours of sleep due to the two forms. What real trust, however, can be put upon a correlation due to 10 pairs? We need, further, some more definite demonstration of how (ϵ) applies to this case with $\bar{u} - \bar{v} = 0$, which seems to involve the assumption that u and v are drawn *at random from the same population.*

It may be of interest in regard to problems of this sort to exhibit a further example of the use of the Neyman-Pearson test as given by them on p. 206 of their paper cited above.

Illustration. A piece of work is carried out by one set of 30 workmen according to Method I, and by a *second* set of 40 workmen according to Method II. The two sets of workmen are supposed of like ability. The resulting frequencies were:

Time in seconds	50	51	52	53	54	55	56	57	58	59	60	Totals
Method I	1	3	5	4	7	5	3	1	1	—	—	$30 = n_1$
Method II	—	1	2	5	8	9	6	3	3	1	2	$40 = n_2$

Here for I, $m_u = 53{\cdot}700$ secs., $s_1 = 1{\cdot}882$ secs.

„ II, $m_v = 55{\cdot}175$ secs., $s_2 = 2{\cdot}072$ secs.

Now according to the test we take

$$x' = \frac{m_u - m_v}{\sqrt{n_1 s_1^2 + n_2 s_2^2}} \sqrt{\frac{n_1 n_2}{n_1 + n_2}} = - \cdot 3663,$$

$$x'^2 = \cdot 1342 \quad \text{and} \quad x = \frac{\cdot 1342}{1 \cdot 1342} = \cdot 11832,$$

$$n = n_1 + n_2 - 1 = 69.$$

This lies outside our Table I for n, but the probability integral is

$$\tfrac{1}{2}(1 + I_x(\tfrac{1}{2}, 34)),$$

and found from the Tables of the Incomplete B-function * = ·998,2199, which agrees with the value ·9982 given by Neyman and Pearson. Thus the odds are about 277 to 1 that if the two samples were from the same population x' would lie outside the limits ± ·3662.

It is clear that such a problem cannot be solved directly by "Student's" ratio as originally given, unless we have the two samples of equal size. In any real case this would be likely to occur, for to produce equal ability in the two samples, the same men would probably be used for both methods. But if this were so, correlation would almost certainly come in and the Neyman-Pearson test would be inapplicable. Hence it becomes all the more important to be certain that "Student's" test can be safely used, when the two populations are correlated member for member.

It appears to me that in applying his test "Student" has really to face two problems, which cannot be solved by a single investigation in the manner he proposes:

(i) If we take two *wholly independent* sets of individuals, and administer the laevo- form of the soporific to one and the dextro- to the other, is there a probability, and what value has it, that the two means differ, and so can we determine which is the more efficient?

(ii) If we administer both soporifics to the *same* set of individuals, i.e. allowing for the individual reactions to the two forms of the drug, will the data indicate that the one is more effective than the other?

Now (i) can be answered by "Student's" test, because he can suppose the samples drawn from the same population, and thus see how improbable the results are. Or, Neyman and Pearson's test may be used, if the samples are of different sizes.

But (i) must be answered before (ii). If (i) show there to be no substantial difference in the hours of sleep of the two sets, then $\bar{u}$ may be put $= \bar{v}$ in (ii). But if the answer to (i) is that $\bar{u}$ and $\bar{v}$ in all probability differ, then it does not seem valid to put $\bar{u} = \bar{v}$ in (ii). It is clear that if $\bar{u}$ be not equal to $\bar{v}$, then a very different value and a much smaller value will be obtained for x' than that given by "Student." The problem thus raised appears to repeat itself in others of "Student's" illustrations, and my object is to press for caution in the application of his test, and indeed in other tests similar to it.

* For most practical purposes, it is adequate to use here the normal curve with standard deviation $1/\sqrt{n-3}$, the standard deviation of the x' curve. Now $x' = -\cdot3662$ and $1/\sqrt{n-3} = \cdot1231$, therefore $x'/\sigma_{x'} = 2\cdot975$, and the corresponding probability ·99853, which for most practical inferences is as serviceable as the correct value ·99822 obtained from the B-function tables.

APPENDIX.

On Interpolation into Table I for small values of $q = \frac{1}{2}(n-1)$.

Interpolation for $q = \frac{1}{2}(n-1)$ is bound to be laborious, even if it be straightforward. In interpolating for q into Table I, it will be found best, particularly in the earlier part of the table, to use a forward difference formula, e.g.

$$u_0(\theta) = u_0 + \theta\Delta u_0 - \frac{\theta(1-\theta)}{2!}\Delta^2 u_0 + \frac{\theta(1-\theta)(2-\theta)}{3!}\Delta^3 u_0 - \frac{\theta(1-\theta)(2-\theta)(3-\theta)}{4!}\Delta^4 u_0 + \ldots.$$

If we use the tabled value $P_x(\frac{1}{2}, q)$, we may have to find, even at $x = \cdot25$, eight or nine differences to get the correct result to seven decimal places. But if we reduce the $P_x(\frac{1}{2}, q)$ to $B_x(\frac{1}{2}, q)$ by the relation $B_x(\frac{1}{2}, q) = \{2P_x(\frac{1}{2}, q) - 1\} \times B(\frac{1}{2}, q)$ four or five differences will suffice for 7-figure accuracy when $x = \cdot25$. For $x = \cdot50$, the seventh difference is required for the $B_x(\frac{1}{2}, q)$'s. The $I_x(\frac{1}{2}, q)$'s would need far more. In order that Δq may not exceed $\cdot25$, we can use when it appears desirable a negative interpolation.

The value of $B(\frac{1}{2}, q)$ is given at the top of each column to assist the reader in reducing the tabled entries to $B_x(\frac{1}{2}, q)$. From the interpolated value of the latter we find $P_x(\frac{1}{2}, q)$ by determining from a table of the complete Γ-functions the complete B-function corresponding to the interpolated value.

Illustrations.

(i) Find the value of $P_{\cdot25}(\frac{1}{2}, 3\cdot25)$.

(*a*) Let us work first with $I_x(\frac{1}{2}, q) = 2P_x(\frac{1}{2}, q) - 1$:

q	$I_{\cdot25}(\frac{1}{2}, q)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7	Δ
3	·792,9688,								
3·5	·829,5293,	·036,5605,							
4	·858,8867,	·029,3574,	−·007,2031,						
4·5	·882,6932,	·023,8065,	−·005,5509,	·001,6522,					
5	·902,1454,	·019,4522,	−·004,3543,	·001,1966,	−·000,4556,				
5·5	·918,1358,	·015,9904,	−·003,4618,	·000,8925,	−·000,3041,	·000,1515,			
6	·931,3450,	·013,2092,	−·002,7812,	·000,6806,	−·000,2119,	·000,0922,	−·000,0593,		
6·5	·942,3012,	·010,9562,	−·002,2530,	·000,5282,	−·000,1524,	·000,0595,	−·000,0327,	·000,0266,	
7	·951,4197,	·009,1185,	−·001,8377,	·000,4153,	−·000,1129,	·000,0395,	−·000,0200,	·000,0127,	−·000,0139.

Here we must go as far as Δ^8.

$$I_{\cdot25}(\tfrac{1}{2}, 3\cdot25) = \cdot792,9688 + \tfrac{1}{2}(\cdot036,5605) - \tfrac{1}{8}(-\cdot007,2031) + \tfrac{1}{16}(\cdot001,6522) - \tfrac{5}{128}(-\cdot000,4556) + \tfrac{7}{256}(\cdot000,1515) - \tfrac{21}{1024}(-\cdot000,0593) + \tfrac{33}{2048}(\cdot000,0266) - \tfrac{429}{32768}(-\cdot000,0139)$$

$$= \cdot792,9688 + \cdot018,2803 + \cdot000,9004 + \cdot000,1033 + \cdot000,0178 + \cdot000,0042 + \cdot000,0012 + \cdot000,0004 + \cdot000,0002$$

$$= \cdot792,9688 + \cdot019\cdot3078 = \cdot812,2766,$$

and $P_{\cdot25}(\frac{1}{2}, 3\cdot25) = \cdot906,1383$, which is accurate to the last figure.

The process is somewhat lengthy and can be shortened by using $B_x(\frac{1}{2}, q)$.

(*b*) Starting from the $I_x(\frac{1}{2}, q)$'s, multiply them by their respective $B(\frac{1}{2}, q)$'s and we obtain the following series:

q	$B_x(\frac{1}{2}, q)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
3	·845,8334,					
3·5	·814,3885,	− ·031,4449,				
4	·785,2678,	− ·029,1207,	·002,3242,			
4·5	·758,2593,	− ·027,0085,	·002,1122,	− ·000,2120,		
5	·733,1721,	− ·025,0872,	·001,9213,	− ·000,1909,	+ ·000·0211,	
5·5	·709,8349,	− ·023,3372,	·001,7500,	− ·000,1713,	+ ·000,0196,	− ·000,0015.

The differencing here is briefer and more effective.

$$B_{\cdot 25}(\tfrac{1}{2}, 3{\cdot}25) = {\cdot}845,8334 - {\cdot}015,7224(5) - {\cdot}000,2905(3) - {\cdot}000,0132(5) - {\cdot}000,0008(2) - {\cdot}000,0000(4)$$
$$= {\cdot}845,8334 - {\cdot}016,0271 = {\cdot}829,8063.$$

But $B(\frac{1}{2}, 3{\cdot}25) = 1{\cdot}021,58087$, hence

$$I_{\cdot 25}(\tfrac{1}{2}, 3{\cdot}25) = B_{\cdot 25}(\tfrac{1}{2}, 3{\cdot}25)/B(\tfrac{1}{2}, 3{\cdot}25) = {\cdot}812,27668$$

and $$P_{\cdot 25}(\tfrac{1}{2}, 3{\cdot}25) = \tfrac{1}{2}\{1 + I_{\cdot 25}(\tfrac{1}{2}, 3{\cdot}25)\} = {\cdot}906,1383,$$

the correct value, as before.

(ii) Find the value of $P_{\cdot 50}(\frac{1}{2}, 3{\cdot}25)$.

Here, even using the B_x's, we must go as far as Δ^7 to be accurate to the seventh figure. Our scheme is as follows:

	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7
$B_{\cdot 50}(\frac{1}{2}, 3\ )$ 1·013,5197,							
$B_{\cdot 50}(\frac{1}{2}, 3{\cdot}5)$ ·949,2072,	− ·064,3125,						
$B_{\cdot 50}(\frac{1}{2}, 4\ )$ ·893,9850,	− ·055,2222,	·009,0903,					
$B_{\cdot 50}(\frac{1}{2}, 4{\cdot}5)$ ·846,1813,	− ·047,8037,	·007,4185,	− ·001,6718,				
$B_{\cdot 50}(\frac{1}{2}, 5\ )$ ·804,4742,	− ·041,7071,	·006,0966,	− ·001,3219,	·000,3499,			
$B_{\cdot 50}(\frac{1}{2}, 5{\cdot}5)$ ·767,8131,	− ·036,6611,	·005,0460,	− ·001,0506,	·000,2713,	− ·000,0786,		
$B_{\cdot 50}(\frac{1}{2}, 6\ )$ ·735,3579,	− ·032,4552,	·004,2059,	− ·000,8401,	·000,2105,	− ·000,0608,	·000,0178,	
$B_{\cdot 50}(\frac{1}{2}, 6{\cdot}5)$ ·706,4329,	− ·028,9250,	·003,5302,	− ·000,6757,	·000,1644,	− ·000,0461,	·000,0147,	− ·000,0031.

Substituting these results in the forward difference formula, we have

$$B_{\cdot 50}(\tfrac{1}{2}, 3{\cdot}25) = 1{\cdot}013,5197 - \tfrac{1}{2}({\cdot}064,3125) - \tfrac{1}{8}({\cdot}009,0903) - \tfrac{1}{16}({\cdot}001,6718) - \tfrac{5}{128}({\cdot}000,3499) - \tfrac{7}{256}({\cdot}000,0786) - \tfrac{21}{1024}({\cdot}000{\cdot}0178) - \tfrac{33}{2048}({\cdot}000,0031)$$
$$= 1{\cdot}013,5197 - {\cdot}032,1562(5) - {\cdot}001,1362(9) - {\cdot}000,1044(8) - {\cdot}000,0136(7) - {\cdot}000,0021(5) - {\cdot}000,0003(7) - {\cdot}000,0000(5)$$
$$= {\cdot}980,1064, \text{ and again } B(\tfrac{1}{2}, 3{\cdot}25) = 1{\cdot}021,58087.$$

Hence $$I_{\cdot 50}(\tfrac{1}{2}, 3{\cdot}25) = {\cdot}959,4016(7),$$

and thus $$P_{\cdot 50}(\tfrac{1}{2}, 3{\cdot}25) = {\cdot}979,7008(3),$$

which is the correct value to seven figures.

(iii) Find the value of $P_{\cdot 10}(\frac{1}{2}, 3\cdot 25)$.

Now $P_{\cdot 10}(\frac{1}{2}, 3\cdot 25)$ is easy to find; we have the following series of differences for $B_{\cdot 10}(\frac{1}{2}, q)$:

q	$B_{\cdot 10}(\frac{1}{2}, q)$	Δ	Δ^2	Δ^3	Δ^4
3	·591,5567,				
3·5	·582,0941,	−·009,4626,			
4	·572,9144,	−·009,1797,	+·000,2829,		
4·5	·564,0073,	−·008,9071,	+·000,2726,	−·000,0103,	
5	·555,3632,	−·008,6441,	+·000,2630,	−·000,0096,	+·000,0007.

Hence
$$B_{\cdot 10}(\tfrac{1}{2}, 3\cdot 25) = \cdot 591,5567 - \cdot 004,73130 - \cdot 000,03536 - \cdot 000,00064 - \cdot 000,00003$$
$$= \cdot 591,5567 - \cdot 004,7673$$
$$= \cdot 586,7894.$$

And
$$I_{\cdot 10}(\tfrac{1}{2}, 3\cdot 25) = B_{\cdot 10}(\tfrac{1}{2}, 3\cdot 25)/B(\tfrac{1}{2}, 3\cdot 25)$$
$$= \cdot 574,3935.$$

Thus
$$P_{\cdot 10}(\tfrac{1}{2}, 3\cdot 25) = \cdot 787,19675,$$

which is exact.

Beyond $x = \cdot 75$ the forward difference method will still apply, but the number of differences required is excessive, if we start with $q = 2$ or 3. For $q = 4$ the eighth difference suffices for 7-figure accuracy; for $q = 5$ the fifth difference will suffice for like accuracy, and so on; this supposes working with $B_x(\frac{1}{2}, q)$ instead of $I_x(\frac{1}{2}, q)$. Thus by the time we get to $q = 8$, there is no trouble. For many statistical purposes four or five figure accuracy is adequate, and accordingly there is less trouble with forward difference work.

For such an extreme case as $I_{\cdot 90}(\frac{1}{2}, 3\cdot 25)$ the limiting difference that the present table provides is the twelfth. Even if we use this and the forward difference formula we shall be out slightly more than unity in the fifth decimal place. If we proceed also to the twelfth difference, using $B_{\cdot 90}(\frac{1}{2}, 3\cdot 25)$, we shall be out less than unity in the sixth decimal place, and assuming the thirteenth difference to be about half the twelfth (as it must be here) we can obtain a value differing from the true value by less than five units in the seventh decimal place. The labour, if straightforward, is considerable, and some will prefer to obtain the result by expansion methods rather than by using the present table of $I_x(\frac{1}{2}, q)$ or $B_x(\frac{1}{2}, q)$ when x approaches unity and q is fractional and small.

TABLE I. *Values of* $P_x(n)$.

$n=$ / $\frac{1}{2}(n-1)=$		2 0·5	3 1·0	4 1·5	5 2·0	6 2·5	7 3·0	8 3·5
$B(\frac{1}{2},\frac{1}{2}(n-1))=$		3·1415,9265+	2·0000,0000	1·5707,9633	1·3333,3333	1·1780,9725−	1·0666,6667	·9817,4770
x	z^2							
·00	·00000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000
·01	·01010	·531,8843	·550,0000	·563,5557	·574,7500e	·584,4589	·593,1269	·601,0142
·02	·02041	·545,1672	·570,7107	·589,7306	·605,3589	·618,8454	·630,8254	·641,6713
·03	·03093	·555,4123	·586,6025+	·609,7119	·628,6048	·644,8257	·659,1614	·672,0739
·04	·04167	·564,0942	·600,0000	·626,4700	·648,0000	·666,3904	·682,5600	·697,0494
·05	·05263	·571,7831	·611,8034	·641,1572	·664,9100	·685,0941	·702,7485−	·718,4861
·06	·06383	·578,7712	·622,4745−	·654,3656	·680,0375−	·701,7381	·720,6194	·737,3622
·07	·07527	·585,2317	·632,2876	·666,4475+	·693,8013	·716,8013	·736,7071	·754,2646
·08	·08696	·591,2774	·641,4214	·677,6328	·706,4752	·730,5973	·751,3623	·769,5793
·09	·09890	·596,9867	·650,0000	·688,0812	·718,2500e	·743,3452	·764,8306	·783,5773
·10	·11111	·602,4164	·658,1139	·697,9093	·729,2651	·755,2051	·777,2922	·796,4581
·11	·12360	·607,6095+	·665,8312	·707,2054	·739,6261	·766,2990	·788,8842	·808,3736
·12	·13636	·612,5995−	·673,2051	·716,0379	·749,4153	·776,7218	·799,7141	·819,4433
·13	·14943	·617,4127	·680,2776	·724,4614	·758,6983	·786,5497	·809,8678	·829,7631
·14	·16279	·622,0709	·687,0829	·732,5203	·767,5285+	·795,8446	·819,4159	·839,4117
·15	·17647	·626,5917	·693,6492	·740,2510	·775,9501	·804,6580	·828,4168	·848,4547
·16	·19048	·630,9899	·700,0000	·747,6842	·784,0000	·813,0330	·836,9200	·856,9474
·17	·20482	·635,2781	·706,1553	·754,8458	·791,7097	·821,0065+	·844,9674	·864,9373
·18	·21951	·639,4672	·712,1320	·761,7578	·799,1062	·828,6101	·852,5953	·872,4651
·19	·23457	·643,5663	·717,9449	·768,4395+	·806,2127	·835,8711	·859,8353	·879,5668
·20	·25000e	·647,5836	·723,6068	·774,9076	·813,0495+	·842,8137	·866,7151	·886,2736
·21	·26582	·651,5263	·729,1288	·781,1765+	·819,6347	·849,4585+	·873,2594	·892,6135−
·22	·28205+	·655,4006	·734,5208	·787,2593	·825,9839	·855,8258	·879,4898	·898,6113
·23	·29870	·659,2121	·739,7916	·793,1673	·832,1113	·861,9309	·885,4260	·904,2893
·24	·31579	·662,9660	·744,9490	·798,9108	·838,0296	·867,7894	·891,0855+	·909,6677
·25	·33333	·666,6667	·750,0000	·804,4989	·843,7500e	·873,4150+	·896,4844	·914,7647
·26	·35135+	·670,3183	·754,9510	·809,9399	·849,2828	·878,8199	·901,6370	·919,5969
·27	·36986	·673,9247	·759,8076	·815,2414	·854,6374	·884,0155+	·906,5567	·924,1796
·28	·38889	·677,4892	·764,5751	·820,4100	·859,8222	·889,0120	·911,2556	·928,5267
·29	·40845+	·681,0150+	·769,2582	·825,4520	·864,8449	·893,8188	·915,7448	·932,6512
·30	·42857	·684,5051	·773,8613	·830,3730	·869,7127	·898,4447	·920,0347	·936,5648
·31	·44928	·687,9620	·778,3882	·835,1782	·874,4322	·902,8976	·924,1349	·940,2787
·32	·47059	·691,3883	·782,8427	·839,8723	·879,0092	·907,1850+	·928,0542	·943,8032
·33	·49254	·694,7865−	·787,2281	·844,4598	·883,4496	·911,3139	·931,8008	·947,1477
·34	·51515+	·698,1585+	·791,5476	·848,9447	·887,7583	·915,2906	·935,3826	·950,3213
·35	·53846	·701,5067	·795,8040	·853,3308	·891,9403	·919,1213	·938,8067	·953,3323
·36	·56250e	·704,8328	·800,0000	·857,6215+	·896,0000	·922,8114	·942,0800	·956,1886
·37	·58730	·708,1387	·804,1381	·861,8201	·899,9416	·926,3663	·945,2088	·958,8976
·38	·61290	·711,4263	·808,2207	·865,9296	·903,7691	·929,7909	·948,1991	·961,4662
·39	·63934	·714,6971	·812,2499	·869,9528	·907,4861	·933,0900	·951,0567	·963,9010
·40	·66667	·717,9529	·816,2278	·873,8923	·911,0961	·936,2680	·953,7868	·966,2084
·41	·69492	·721,1951	·820,1562	·877,7505+	·914,6023	·939,3290	·956,3947	·968,3940
·42	·72414	·724,4253	·824,0370	·881,5298	·918,0078	·942,2769	·958,8850+	·970,4636
·43	·75439	·727,6449	·827,8719	·885,2324	·921,3154	·945,1156	·961,2625+	·972,4224
·44	·78571	·730,8553	·831,6625−	·888,8601	·924,5280	·947,8486	·963,5315−	·974,2755−
·45	·81818	·734,0579	·835,4102	·892,4150+	·927,6480	·950,4793	·965,6961	·976,0276
·46	·85185+	·737,2540	·839,1165−	·895,8988	·930,6780	·953,0110	·967,7603	·977,6834
·47	·88679	·740,4450−	·842,7827	·899,3132	·933,6202	·955,4466	·969,7280	·979,2472
·48	·92308	·743,6321	·846,4102	·902,6597	·936,4768	·957,7892	·971,6028	·980,7231
·49	·96078	·746,8167	·850,0000	·905,9398	·939,2500e	·960,0417	·973,3881	·982,1152
·50	1·00000	·750,0000	·853,5534	·909,1549	·941,9417	·962,2061	·975,0874	·983,4272

TABLE I (*continued*).

$n=$ / $\frac{1}{2}(n-1)=$		2 0·5	3 1·0	4 1·5	5 2·0	6 2·5	7 3·0	8 3·5
$B(\frac{1}{2},\frac{1}{2}(n-1))=$		3·1415,9265+	2·0000,0000	1·5707,9633	1·3333,3333	1·1780,9725−	1·0666,6667	·9817,4770
x	z^2							
·50	1·0000	·750,000	·853,5534	·909,1549	·941,9417	·962,2061	·975,0874	·983,4272
·51	1·04081	·753,1833	·857,0714	·912,3064	·944,5539	·964,2866	·976,7037	·984,6629
·52	1·08333	·756,3679	·860,5551	·915,3955−	·947,0884	·966,2843	·978,2403	·985,8256
·53	1·12766	·759,5550+	·864,0055−	·918,4232	·949,5468	·968,2019	·979,7001	·986,9187
·54	1·17391	·762,7460	·867,4235−	·921,3908	·951,9309	·970,0419	·981,0859	·987,9455+
·55	1·22222	·765,9421	·870,8099	·924,2993	·954,2422	·971,8065−	·982,4005+	·988,9090
·56	1·27273	·769,1447	·874,1657	·927,1496	·956,4822	·973,4977	·983,6466	·289,8122
·57	1·32558	·772,3551	·877,4917	·929,9426	·958,6524	·975,1177	·984,8268	·990,6580
·58	1·38095+	·775,5747	·880,7887	·932,6793	·960,7543	·976,6685+	·985,9434	·991,4489
·59	1·43902	·778,8049	·884,0573	·935,3603	·962,7890	·978,1521	·986,9990	·992,1878
·60	1·50000	·782,0471	·887,2983	·937,9865−	·964,7580	·979,5703	·987,9959	·992,8771
·61	1·56410	·785,3029	·890,5125−	·940,5585−	·966,6624	·980,9250−	·988,9363	·993,5193
·62	1·63158	·788,5737	·893,7004	·943,0770	·968,5035−	·982,2179	·989,8223	·994,1167
·63	1·70270	·791,8613	·896,8627	·945,5427	·970,2823	·983,4507	·990,6562	·994,6715+
·64	1·77778	·795,1672	·900,0000	·947,9560	·972,0000	·984,6253	·991,4400	·995,1860
·65	1·85714	·798,4933	·903,1129	·950,3175−	·973,6576	·985,7431	·992,1756	·995,6623
·66	1·94118	·801,8415−	·906,2019	·952,6276	·975,2562	·986,8058	·992,8651	·996,1023
·67	2·03030	·805,2135+	·909,2676	·954,8869	·976,7968	·987,8150+	·993,5103	·996,5081
·68	2·12500	·808,6117	·912,3106	·957,0956	·978,2803	·988,7722	·994,1130	·996,8814
·69	2·22581	·812,0380	·915,3312	·959,2542	·979,7075+	·989,6789	·994,6750+	·997,2242
·70	2·33333	·815,4949	·918,3300	·961,3629	·981,0795+	·990,5364	·995,1982	·997,5381
·71	2·44828	·818,9850−	·921,3075−	·963,4219	·982,3971	·991,3464	·995,6841	·997,8249
·72	2·57143	·822,5108	·924,2641	·965,4316	·983,6610	·992,1101	·996,1344	·998,0861
·73	2·70370	·826,0753	·927,2002	·967,3920	·984,8722	·992,8290	·996,5508	·998,3234
·74	2·84615+	·829,6817	·930,1163	·969,3033	·986,0314	·993,5044	·996,9348	·998,5382
·75	3·00000	·833,3333	·933,0127	·971,1656	·987,1393	·994,1376	·997,2880	·998,7320
·76	3·16667	·837,0340	·935,8899	·972,9788	·988,1967	·994,7300	·997,6119	·998,9062
·77	3·34783	·840,7879	·938,7482	·974,7430	·989,2043	·995,2828	·997,9079	·999,0621
·78	3·54545+	·844,5994	·941,5880	·976,4581	·990,1627	·995,7974	·998,1776	·999,2011
·79	3·76190	·848,4737	·944,4097	·978,1240	·991,0727	·996,2750+	·998,4222	·999,3244
·80	4·00000	·852,4164	·947,2136	·979,7403	·991,9350−	·996,7169	998,6432	·999,4331
·81	4·26316	·856,4337	·950,0000	·981,3070	·992,7500e	·997,1242	·998,8419	·999,5285−
·82	4·55556	·860,5328	·952,7693	·982,8235−	·993,5185−	·997,4984	·999,0196	·999,6116
·83	4·88235+	·864,7219	·955,5217	·984,2895+	·994,2410	·997,8405+	·999,1777	·999,6834
·84	5·25000	·869,0101	·958,2576	·985,7045−	·994,9182	·998,1518	·999,3174	·999,7451
·85	5·66667	·873,4083	·960,9772	·987,0677	·995,5505+	·998,4336	·999,4400	·999,7976
·86	6·14286	·877,9291	·963,6809	·988,3785+	·996,1386	·998,6871	·999,5466	·999,8417
·87	6·69231	·882,5873	·966,3690	·989,6360	·996,6829	·998,9135+	·999,6385+	·999,8784
·88	7·33333	·887,4005+	·969,0416	·990,8390	·997,1841	·999,1141	·999,7169	·999,9085+
·89	8·09091	·892,3905−	·971,6991	·991,9864	·997,6425+	·999,2901	·999,7828	·999,9328
·90	9·00000	·897,5836	·974,3416	·993,0766	·998,0587	·999,4428	·999,8375+	·999,9521
·91	10·11111	·903,0133	·976,9696	·994,1078	·998,4332	·999,5735−	·999,8820	·999,9670
·92	11·50000	·908,7226	·979,5832	·995,0779	·998,7665−	·999,6835−	·999,9175−	·999,9782
·93	13·28571	·914,7683	·982,1825+	·995,9841	·999,0589	·999,7742	·999,9449	·999,9864
·94	15·66667	·921,2288	·984,7680	·996,8232	·999,3110	·999,8470	·999,9655−	·999,9921
·95	19·00000	·928,2169	·987,3397	·997,5909	·999,5232	·999,9034	·999,9801	·999,9959
·96	24·00000	·935,9058	·989,8979	·998,2815+	·999,6959	·999,9449	·999,9898	·999,9981
·97	32·33333	·944,5877	·992,4429	·998,8873	·999,8295+	·999,9732	·999,9957	·999,9993
·98	49·00000	·954,8328	·994,9747	·999,3961	·999,9245−	·999,9903	·999,9987	·999,9998
·99	99·00000	·968,1157	·997,4937	·999,7872	·999,9812	·999,9983	·999,9998	1·000,0000
1·00	∞	1·000,0000	1·000,0000	1·000,0000	1·000,0000	1·000,0000	1·000,0000	

TABLE I (*continued*).

$n=$ / $\frac{1}{2}(n-1)=$		9 / 4·0	10 / 4·5	11 / 5·0	12 / 5·5	13 / 6·0	14 / 6·5	15 / 7·0
$B(\frac{1}{2},\frac{1}{2}(n-1))=$		·9142,8571	·8590,2924	·8126,9841	·7731,2632	·7388,1674	·7086,9912	·6819,8468
x	z^2							
·00	·00000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000
·01	·01010	·608,2878	·615,0625$^+$	·621,4209	·627,4250$^+$	·633,1226	·638,5513	·643,7418
·02	·02041	·651,6230	·660,8451	·669,4569	·677,5476	·685,1864	·692,4281	·699,3168
·03	·03093	·683,8613	·694,7289	·704,8254	·714,2626	·723,1270	·731,4877	·739,4002
·04	·04167	·710,2080	·722,2773	·733,4323	·743,8051	·753,4981	·762,5930	·771,1560
·05	·05263	·732,7039	·745,6768	·757,6044	·768,6378	·778,8943	·788,4677	·797,4342
·06	·06383	·752,4086	·766,0651	·778,5552	·790,0479	·800,6752	·810,5424	·819,7353
·07	·07527	·769,9591	·784,1281	·797,0179	·808,8153	·819,6662	·829,6872	·838,9738
·08	·08696	·785,7758	·800,3193	·813,4786	·825,4578	·836,4166	·846,4828	·855,7609
·09	·09890	·800,1543	·814,9584	·828,2807	·840,3423	·851,3163	·861,3416	·870,5318
·10	·11111	·813,3125$^+$	·828,2818	·841,6785$^+$	·853,7408	·864,6550$^+$	·874,5708	·883,6106
·11	·12360	·825,4173	·840,4705$^+$	·853,8675$^-$	·865,8628	·876,6560	·886,4074	·895,2477
·12	·13636	·836,5998	·851,6675$^-$	·865,0019	·876,8740	·887,4963	·897,0391	·905,6418
·13	·14943	·846,9657	·861,9880	·875,2065$^+$	·886,9085$^+$	·897,3191	·906,6184	·914,9538
·14	·16279	·856,6019	·871,5270	·884,5844	·896,0772	·906,2428	·915,2711	·923,3169
·15	·17647	·865,5809	·880,3638	·893,2216	·904,4729	·914,3668	·923,1026	·930,8424
·16	·19048	·873,9640	·888,5658	·901,1913	·912,1742	·921,7752	·930,2024	·937,6248
·17	·20482	·881,8039	·896,1908^5	·908,5564	·919,2491	·928,5406	·936,6475$^+$	·943,7452
·18	·21951	·889,1461	·903,2890	·915,3714	·925,7561	·934,7256	·942,5044	·949,2736
·19	·23457	·896,0306	·909,9040	·921,6840	·931,7469	·940,3853	·947,8311	·954,2710
·20	·25000^e	·902,4922	·916,0747	·927,5362	·937,2666	·945,5678	·952,6788	·958,7911
·21	·26582	·908,5623	·921,8353	·932,9655$^-$	·942,3554	·950,3161	·957,0926	·962,8809
·22	·28205$^+$	·914,2687	·927,2165$^-$	·938,0052	·947,0494	·954,6683	·961,1127	·966,5824
·23	·29870	·919,6362	·932,2459	·942,6854	·951,3806	·958,6584	·964,7748^5	·969,9328
·24	·31579	·924,6876	·936,9484	·947,0330	·955,3780	·962,3173	·968,1112	·972,9653
·25	·33333	·929,4434	·941,3466	·951,0727	·959,0679	·965,6725$^-$	·971,1506	·975,7099
·26	·35135$^+$	·933,9221	·945,4611	·954,8267	·962,4741	·968,7491	·973,9191	·978,1932
·27	·36986	·938,1410	·949,3108	·958,3154	·965,6182	·971,5700	·976,4404	·980,4395$^+$
·28	·38889	·942,1156	·952,9130	·961,5575$^-$	·968,5201	·974,1558	·978,7358	·982,4706
·29	·40845$^+$	·945,8606	·956,2834	·964,5700	·971,1980	·976,5253	·980,8247	·984,3063
·30	·42857	·949,3892	·959,4369	·967,3689	·973,6684	·978,6960	·982,7249	·985,9643
·31	·44928	·952,7140	·962,3870	·969,9686	·975,9467	·980,6837	·984,4524	·987,4610
·32	·47059	·955,8463	·965,1463	·972,3826	·978,0471	·982,5028	·986,0221	·988,8111
·33	·49254	·958,7969	·967,7266	·974,6234	·979,9824	·984,1668	·987,4473	·990,0280
·34	·51515$^+$	·961,5760	·970,1386	·976,7026	·981,7648	·985,6879	·988,7405$^+$	·991,1239
·35	·53846	·964,1927	·972,3927	·978,6310	·983,4054	·987,0774	·989,9129	·992,1100
·36	·56250^e	·966,6560	·974,4984	·980,4186	·984,9146	·988,3458	·990,9750$^-$	·992,9964
·37	·58730	·968,9741	·976,4644	·982,0747	·986,3019	·989,5027	·991,9361	·993,7924
·38	·61290	·971,1546	·978,2993	·983,6080	·987,5762	·990,5570	·992,8050$^+$	·994,5063
·39	·63934	·973,2051	·980,0108	·985,0268	·988,7459	·991,5169	·993,5898	·995,1459
·40	·66667	·975,1322	·981,6063	·986,3385$^+$	·989,8185$^-$	·992,3900	·994,2979	·995,7182
·41	·69492	·976,9426	·983,0926	·987,5505$^-$	·990,8012	·993,1833	·994,9358	·996,2296
·42	·72414	·978,6424	·984,4764	·988,6692	·991,7008	·993,9033	·995,5100	·996,6860
·43	·75439	·980,2374	·985,7637	·989,7011	·992,5233	·994,5560	·996,0260	·997,0927
·44	·78571	·981,7331	·986,9604	·990,6519	·993,2746	·995,1470	·996,4891	·997,4545$^-$
·45	·81818	·983,1348	·988,0718	·991,5272	·993,9601	·995,6814	·996,9042	·997,7758
·46	·85185$^+$	·984,4474	·989,1032	·992,3321	·994,5847	·996,1640	·997,2756	·998,0608
·47	·88679	·985,6757	·990,0594	·993,0714	·995,1532	·996,5992	·997,6075$^-$	·998,3131
·48	·92308	·986,8241	·990,9451	·993,7497	·995,6699	·996,9909	·997,9035$^-$	·998,5359
·49	·96078	·987,8968	·991,7645	·994,3713	·996,1389	·997,3431	·998,1670	·998,7325$^-$
·50	1·00000	·988,8980	·992,5218	·994,9402	·996,5638	·997,6592	·998,4011	·998,9054

TABLE I (*continued*).

$n=$ / $\frac{1}{2}(n-1)=$		9 / 4·0	10 / 4·5	11 / 5·0	12 / 5·5	13 / 6·0	14 / 6·5	15 / 7·0
$B(\frac{1}{2},\frac{1}{2}(n-1))=$		·9142,8571	·8590,2924	·8126,9841	·7731,2632	·7388,1674	·7086,9912	·6819,8468
x	z^2							
·50	1·00000	·988,8980	·992,5218	·994,9402	·996,5638	·997,6592	·998,4011	·998,9054
·51	1·04081	·989,8316	·993,2209	·995,4601	·996,9484	·997,9423	·998,6088	·999,0573
·52	1·08333	·990,7011	·993,8654	·995,9346	·997,2958	·998,1955$^+$	·998,7927	·999,1903
·53	1·12766	·991,5101	·994,4589	·996,3670	·997,6091	·998,4215$^-$	·998,9550$^+$	·999,3066
·54	1·17391	·992,2620	·995,0047	·996,7604	·997,8911	·998,6227	·999,0981	·999,4080
·55	1·22222	·992,9599	·995,5057	·997,1177	·998,1444	·998,8016	·999,2239	·999,4962
·56	1·27273	·993,6069	·995,9651	·997,4416	·998,3716	·998,9602	·999,3342	·999,5727
·57	1·32558	·994,2059	·996,3856	·997,7348	·998,5749	·999,1005$^-$	·999,4307	·999,6388
·58	1·38095$^+$	·994,7597	·996,7699	·997,9996	·998,7564	·999,2243	·999,5148	·999,6958
·59	1·43902	·995,2708	·997,1204	·998,2383	·998,9180	·999,3333	·999,5880	·999,7448
·60	1·50000	·995,7419	·997,4395$^-$	·998,4530	·999,0616	·999,4289	·999,6515$^+$	·999,7868
·61	1·56410	·996,1753	·997,7294	·998,6456	·999,1889	·999,5127	·999,7064	·999,8226
·62	1·63158	·996,5733	·997,9923	·998,8180	·999,3014	·999,5857	·999,7536	·999,8531
·63	1·70270	·996,9382	·998,2301	·998,9720	·999,4005$^+$	·999,6492	·999,7942	·999,8789
·64	1·77778	·997,2720	·998,4448	·999,1091	·999,4876	·999,7043	·999,8289	·999,9007
·65	1·85714	·997,5767	·998,6380	·999,2308	·999,5638	·999,7518	·999,8584	·999,9190
·66	1·94118	·997,8543	·998,8116	·999,3386	·999,6304	·999,7927	·999,8835$^-$	·999,9343
·67	2·03030	·998,1065$^-$	·998,9669	·999,4336	·999,6882	·999,8278	·999,9046	·999,9470
·68	2·12500	·998,3350$^+$	·999,1057	·999,5172	·999,7383	·999,8577	·999,9224	·999,9576
·69	2·22581	·998,5416	·999,2291	·999,5904	·999,7815$^+$	·999,8831	·999,9372	·999,9662
·70	2·33333	·998,7278	·999,3385$^+$	·999,6543	·999,8186	·999,9045$^+$	·999,9496	·999,9733
·71	2·44828	·998,8951	·999,4352	·999,7099	·999,8503	·999,9225$^+$	·999,9598	·999,9791
·72	2·57143	·999,0449	·999,5203	·999,7579	·999,8773	·999,9376	·999,9682	·999,9837
·73	2·70370	·999,1785$^-$	·999,5949	·999,7993	·999,9001	·999,9501	·999,9750$^+$	·999,9875$^-$
·74	2·84615$^+$	·999,2972	·999,6600	·999,8347	·999,9193	·999,9604^5	·999,9806	·999,9904
·75	3·00000	·999,4023	·999,7165$^+$	·999,8649	·999,9353	·999,9689	·999,9850$^+$	·999,9928
·76	3·16667	·999,4949	·999,7653	·999,8904	·999,9486	·999,9758	·999,9886	·999,9946
·77	3·34783	·999,5761	·999,8072	·999,9119	·999,9595$^+$	·999,9814	·999,9914	·999,9960
·78	3·54545$^+$	·999,6469	·999,8430	·999,9298	·999,9685$^-$	·999,9858	·999,9936	·999,9971
·79	3·76190	·999,7083	·999,8733	·999,9446	·999,9757	·999,9893	·999,9953	·999,9979
·80	4·00000	·999,7612	·999,8988	·999,9568	·999,9815$^+$	·999,9921	·999,9966	·999,9985$^+$
·81	4·26316	·999,8064	·999,9200	·999,9668	·999,9861	·999,9942	·999,9976	·999,9990
·82	4·55556	·999,8448	·999,9376	·999,9748	·999,9898	·999,9958	·999,9983	·999,9993
·83	4·88235$^+$	·999,8771	·999,9520	·999,9811	·999,9926	·999,9971	·999,9988	·999,9995$^+$
·84	5·25000	·999,9040	·999,9636	·999,9861	·999,9947	·999,9980	·999,9992	·999,9997
·85	5·66667	·999,9262	·999,9729	·999,9900	·999,9963	·999,9986	·999,9995$^-$	·999,9998
·86	6·14286	·999,9443	·999,9802	·999,9930	·999,9975$^-$	·999,9991	·999,9997	·999,9999
·87	6·69231	·999,9587	·999,9859	·999,9952	·999,9983	·999,9994	·999,9998	·999,9999
·88	7·33333	·999,9702	·999,9902	·999,9968	·999,9989	·999,9996	·999,9999	1·000,0000
·89	8·09091	·999,9790	·999,9934	·999,9979	·999,9993	·999,9998	·999,9999	
·90	9·00000	·999,9857	·999,9957	·999,9987	·999,9996	·999,9999	1·000,0000	
·91	10·11111	·999,9907	·999,9974	·999,9992	·999,9998	·999,9999		
·92	11·50000	·999,9942	·999,9984^5	·999,9996	·999,9999	1·000,0000		
·93	13·28571	·999,9966	·999,9992	·999,9998	·999,9999			
·94	15·66667	·999,9982	·999,9996	·999,9999	1·000,0000			
·95	19·00000	·999,9991	·999,9998	1·000,0000				
·96	24·00000	·999,9996	·999,9999					
·97	32·33333	·999,9999	1·000,0000					
·98	49·00000	1·000,0000						
·99	99·00000							
1·00	∞							

TABLE I (*continued*).

$n=$ / $\frac{1}{2}(n-1)=$		16 / 7·5	17 / 8·0	18 / 8·5	19 / 9·0	20 / 9·5	21 / 10·0	22 / 10·5
$B(\frac{1}{2}, \frac{1}{2}(n-1))=$		·6580,7776	·6365,1994	·6169,4790	·5990,7674	·5826,7301[5]	·5675,4639	·5535,3936
x	z^2							
·00	·00000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000
·01	·01010	·648,7191	·653,5039	·658,1140	·662,5644	·666,8679	·671,0359	·675,0782
·02	·02041	·705,8891	·712,1754	·718,2015+	·723,9893	·729,5579	·734,9237	·740,1014
·03	·03093	·746,9108	·754,0578	·760,8738	·767,3869	·773,6213	·779,5979	·785,3355−
·04	·04167	·779,2420	·786,8968	·794,1595−	·801,0635+	·807,6379	·813,9080	·819,8961
·05	·05263	·805,8570	·813,7891	·821,2756	·828,3552	·835,0616	·841,4242	·847,4690
·06	·06383	·828,3252	·836,3720	·843,9267	·851,0331	·857,7294	·864,0489	·870,0211
·07	·07527	·847,6050−	·855,6473	·863,1575+	·870,1845+	·876,7706	·882,9530	·888,7645−
·08	·08696	·864,3378	·872,2865+	·879,6693	·886,5399	·892,9446	·898,9244	·904,5151
·09	·09890	·878,9811	·886,7689	·893,9628	·900,6211	·906,7943	·912,5264	·917,8563
·10	·11111	·891,8758	·899,4520	·906,4120	·912,8183	·918,7250+	·924,1795+	·929,2235−
·11	·12360	·903,2856	·910,6124	·917,3057[5]	·923,4323	·929,0497	·934,2081[5]	·938,9517
·12	·13636	·913,4195−	·920,4692	·926,8732	·932,7019	·938,0160	·942,8686	·947,3056
·13	·14943	·922,4470	·929,2002	·935,2998	·940,8199	·945,8240	·950,3674	·954,4981
·14	·16279	.930,5081	·936,9517	·942,7383	·947,9448	·952,6376	·956,8737	·960,7029
·15	·17647	·937,7198	·943,8463	·949,3160	·954,2088	·958,5931	·962,5276	·966,0635+
·16	·19048	·944,1812	·949,9875−	·955,1406	·959,7231	·963,8050−	·967,4466	·970,7000
·17	·20482	·949,9774	·955,4636	·960,3036	·964,5820	·968,3702	·971,7298	·974,7131
·18	·21951	·955,1815+	·960,3508	·964,8837	·968,8664	·972,3716	·975,4613	·978,1884
·19	·23457	·959,8572	·964,7151	·968,9489	·972,6460	·975,8800	·978,7132	·981,1987
·20	·25000[e]	·964,0602	·968,6140	·972,5583	·975,9812	·978,9569	·981,5476	·983,8063
·21	·26582	·967,8395−	·972,0980	·975,7635−	·978,9245−	·981,6552	·984,0178	·986,0647
·22	·28205+	·971,2383	·975,2116	·978,6098	·981,5217	·984,0213	·986,1701	·988,0201
·23	·29870	·974,2951	·977,9939	·981,1370	·983,8130	·986,0953	·988,0448	·989,7123
·24	·31579	·977,0440	·980,4798	·983,3803	·985,8338	·987,9127	·989,6769	·991,1760
·25	·33333	·979,5155+	·982,7002	·985,3710	·987,6152	·989,5042	·991,0967	·992,4410
·26	·35135+	·981,7370	·984,6826	·987,1365+	·989,1847	·990,8971	·992,3311	·993,5335+
·27	·36986	·983,7329	·986,4518	·988,7015−	·990,5665−	·992,1152	·993,4033	·994,4761
·28	·38889	·985,5252	·988,0297	·990,0877	·991,7821	·993,1795+	·994,3337	·995,2884
·29	·40845+	·987,1339	·989,4361	·991,3148	·992,8507	·994,1086	·995,1403	·995,4878
·30	·42857	·988,5767	·990,6887	·992,3999	·993,7891	·994,9187	·995,8387	·996,5891
·31	·44928	·989,8698	·991,8033	·993,3587	·994,6123	·995,6244	·996,4428	·997,1054
·32	·47059	·991,0279	·992,7943	·994,2049	·995,3336	·996,2383	·996,9644	·997,5481
·33	·49254	·992,0641	·993,6745+	·994,9511	·995,9650−	·996,7716	·997,4142	·997,9272
·34	·51515+	·992,9903	·994,4555−	·995,6082	·996,5169	·997,2344	·997,8018	·998,2512
·35	·53846	·993,8174	·995,1476	·996,1862	·996,9986	·997,6353	·998,1349	·998,5276
·36	·56250[e]	·994,5552	·995,7602[5]	·996,6938	·997,4185+	·997,9820	·998,4209	·998,7631
·37	·58730	·995,2126	·996,3019	·997,1392	·997,7840	·998,2815+	·998,6659	·998,9633
·38	·61290	·995,7975+	·996,7800	·997,5292	·998,1016	·998,5397	·998,8754	·999,1332
·39	·63934	·996,3174	·997,2015+	·997,8703	·998,3771	·998,7618	·999,0543	·999,2770
·40	·66667	·996,7787	·997,5726	·998,1680	·998,6156	·998,9526	·999,2067	·999,3985+
·41	·69492	·997,1876	·997,8986	·998,4276	·998,8218	·999,1161	·999,3362	·999,5010
·42	·72414	·997,5494	·998,1847	·998,6534	·998,9997	·999,2560	·999,4461	·999,5871
·43	·75439	·997,8689	·998,4353	·998,8494	·999,1528	·999,3754	·999,5390	·999,6594
·44	·78571	·998,1508	·998,6544	·999,0193	·999,2843	·999,4770	·999,6175−	·999,7199
·45	·81818	·998,3989	·998,8455−	·999,1662	·999,3970	·999,5634	·999,6885+	·999,7704
·46	·85185+	·998,6170	·999,0119	·999,2920	·999,4934	·999,6366	·999,7390	·999,8123
·47	·88679	·998,8082	·999,1565+	·999,4021	·999,5756	·999,6984	·999,7854	·999,8472
·48	·92308	·998,9756	·999,2819	·999,4959	·999,6456	·999,7505+	·999,8242	·999,8760
·49	·96078	·999,1217	·999,3904	·999,5762	·999,7050−	·999,7944	·999,8565−	·999,8998
·50	1·00000	·999,2491	·999,4840	·999,6448	·999,7552	·999,8311	·999,8833	·999,9193

TABLE I (*continued*).

$n=$ / $\frac{1}{2}(n-1)=$		16 / 7·5	17 / 8·0	18 / 8·5	19 / 9·0	20 / 9·5	21 / 10·0	22 / 10·5
$B(\frac{1}{2}, \frac{1}{2}(n-1))=$		·6580,7776	·6365,1904	·6169,4790	·5990,7674	·5826,7301[5]	·5675,4639	·5535,3936
x	z^2							
·50	1·00000	·999,2491	·999,4840	·999,6448	·999,7552	·999,8311	·999,8833	·999,9193
·51	1·04081	·999,3599	·999,5645+	·999,7033	·999,7976	·999,8617	·999,9054	·999,9353
·52	1·08333	·999,4559	·999,6337	·999,7530	·999,8332	·999,8872	·999,9237	·999,9483
·53	1·12766	·999,5390	·999,6929	·999,7951	·999,8631	·999,9084	·999,9387	·999,9589
·54	1·17391	·999,6106	·999,7434	·999,8307	·999,8881	·999,9259	·999,9509	·999,9675−
·55	1·22222	·999,6723	·999,7864	·999,8606	·999,9089	·999,9404	·999,9609	·999,9744
·56	1·27273	·999,7252	·999,8229	·999,8857	·999,9261	·999,9522	·999,9690	·999,9799
·57	1·32558	·999,7704	·999,8537	·999,9067	·999,9404	·999,9619	·999,9756	·999,9843
·58	1·38095+	·999,8089	·999,8797	·999,9242	·999,9521	·999,9697	·999,9808	·999,9879
·59	1·43902	·999,8416	·999,9015+	·999,9387	·999,9617	·999,9761	·999,9851	·999,9906
·60	1·50000	·999,8693	·999,9197	·999,9506	·999,9696	·999,9812	·999,9884	·999,9928
·61	1·56410	·999,8927	·999,9349	·999,9605−	·999,9759	·999,9853	·999,9911	·999,9945+
·62	1·63158	·999,9123	·999,9475−	·999,9685+	·999,9811	·999,9886	·999,9932	·999,9959
·63	1·70270	·999,9286	·999,9579	·999,9751	·999,9852	·999,9912	·999,9948	·999,9969
·64	1·77778	·999,9423	·999,9664	·999,9804	·999,9885+	·999,9933	·999,9961	·999,9977
·65	1·85714	·999,9536	·999,9733	·999,9847	·999,9912	·999,9949	·999,9971	·999,9983
·66	1·94118	·999,9629	·999,9790	·999,9881	·999,9932	·999,9962	·999,9978	·999,9988
·67	2·03030	·999,9705+	·999,9836	·999,9908	·999,9949	·999,9971	·999,9984	·999,9991
·68	2·12500	·999,9767	·999,9872	·999,9930	·999,9961	·999,9979	·999,9988	·999,9993
·69	2·22581	·999,9818	·999,9902	·999,9947	·999,9971	·999,9984	·999,9991	·999,9995+
·70	2·33333	·999,9858	·999,9925−	·999,9960	·999,9979	·999,9989	·999,9994	·999,9997
·71	2·44828	·999,9891	·999,9943	·999,9970	·999,9984	·999,9992	·999,9996	·999,9998
·72	2·57143	·999,9917	·999,9957	·999,9978	·999,9989	·999,9994	·999,9997	·999,9998
·73	2·70370	·999,9937	·999,9968	·999,9984	·999,9992	·999,9996	·999,9998	·999,9999
·74	2·84615+	·999,9953	·999,9977	·999,9988	·999,9994	·999,9997	·999,9999	·999,9999
·75	3·00000	·999,9965−	·999,9983	·999,9992	·999,9996	·999,9998	·999,9999	1·000,0000
·76	3·16667	·999,9974	·999,9988	·999,9994	·999,9997	·999,9999	·999,9999	
·77	3·34783	·999,9981	·999,9991	·999,9996	·999,9998	·999,9999	1·000,0000	
·78	3·54545+	·999,9987	·999,9994	·999,9997	·999,9999	·999,9999		
·79	3·76190	·999,9991	·999,9996	·999,9998	·999,9999	1·000,0000		
·80	4·00000	·999,9994	·999,9997	·999,9999	·999,9999			
·81	4·26316	·999,9996	·999,9998	·999,9999	1·000,0000			
·82	4·55556	·999,9997	·999,9999	1·000,0000				
·83	4·88235+	·999,9998	·999,9999					
·84	5·25000	·999,9999	1·000,0000					
·85	5·66667	·999,9999						
·86	6·14286	1·000,0000						
·87	6·69231							
·88	7·33333							
·89	8·09091							
·90	9·00000							
·91	10·11111							
·92	11·50000							
·93	13·28571							
·94	15·66667							
·95	19·00000							
·96	24·00000							
·97	32·33333							
·98	49·00000							
·99	99·00000							
1·00	∞							

TABLE I (*continued*).

$n=$ / $\frac{1}{2}(n-1)=$		23 / 11·0	24 / 11·5	25 / 12·0	26 / 12·5	27 / 13·0	28 / 13·5	29 / 14·0
$B(\frac{1}{2}, \frac{1}{2}(n-1))=$		·5405,2037	·5283,7848	·5170,1948	·5063,6271	·4963,3870	·4868,8722	·4779,5579
x	z^2							
·00	·00000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000	·500,0000
·01	·01010	·679,0034	·682,8193	·686,5326	·690,1497	·693,6760	·697,1166	·700,4760
·02	·02041	·745,1036	·749,9419	·754,6265⁻	·759,1664	·763,5700	·767,8448	·771,9976
·03	·03093	·790,8504	·796,1572	·801,2691	·806,1979	·810,9542	·815,5477	·819,9874
·04	·04167	·825,6221	·831,1037	·836,3566	·841,3951	·846,2322	·850,8797	·855,3482
·05	·05263	·853,2190	·858,6947	·863,9147	·868,8955⁺	·873,6523	·878,1986	·882,5472
·06	·06383	·875,6721	·881,0251	·886,1012	·890,9191	·895,4961	·899,8475⁺	·903,9876
·07	·07527	·894,2340	·899,3876	·904,2485⁻	·908,8376	·913,1738	·917,2745⁻	·921,1552
·08	·08696	·909,7485⁺	·914,6530	·919,2540	·923,5744	·927,6348	·931,4538	·935,0485⁻
·09	·09890	·922,8185⁻	·927,4435⁻	·931,7586	·935,7884	·939,5551	·943,0786	·946,3770
·10	·11111	·933,8934	·938,2221	·942,2386	·945,9688	·949,4362	·952,6619	·955,6650⁻
·11	·12360	·943,3191	·947,3448	·951,0593	·954,4899	·957,6611	·960,5947	·963,3106
·12	·13636	·951,3679	·955,0912	·958,5074	·961,6446	·964,5283	·967,1810	·969,6230
·13	·14943	·958,2584	·961,6852	·964,8115⁺	·967,6662	·970,2752	·972,6615⁻	·974,8457
·14	·16279	·964,1685⁺	·967,3087	·970,1570	·972,7428	·975,0924	·977,2291	·979,1737
·15	·17647	·969,2451	·972,1110	·974,6954	·977,0279	·979,1351	·981,0401	·982,7637
·16	·19048	·973,6101	·976,2160	·978,5520	·980,6480	·982,5303	·984,2219	·985,7435⁻
·17	·20482	·977,3658	·979,7270	·981,8311	·983,7077	·985,3828	·986,8794	·988,2174
·18	·21951	·980,5987	·982,7311	·984,6198	·986,2942	·987,7798	·989,0990	·990,2713
·19	·23457	·983,3819	·985,3017	·986,9917	·988,4807	·989,7937	·990,9526	·991,9761
·20	·25000[c]	·985,7780	·987,5011	·989,0085⁺	·990,3284	·991,4852	·992,4998	·993,3904
·21	·26582	·987,8403	·989,3823	·990,7228	·991,8893	·992,9051	·993,7906	·994,5629
·22	·28205⁺	·989,6146	·990,9906	·992,1792	·993,2069	·994,0963	·994,8665⁻	·995,5340
·23	·29870	·991,1404	·992,3648	·993,4156	·994,3184	·995,0946	·995,7625⁻	·996,3376
·24	·31579	·992,4515⁺	·993,5380	·994,4644	·995,2551	·995,9305⁻	·996,5079	·997,0018
·25	·33333	·993,5773	·994,5388	·995,3532	·996,0436	·996,6346	·997,1271	·997,5500⁺
·26	·35135⁺	·994,5430	·995,3915⁺	·996,1055⁻	·996,7067	·997,2135⁻	·997,6410	·998,0019
·27	·36986	·995,3706	·996,1174	·996,7415⁺	·997,2635⁺	·997,7006	·998,0667	·998,3737
·28	·38889	·996,0791	·996,7345⁺	·997,2786	·997,7305⁻	·998,1062	·998,4189	·998,6792
·29	·40845⁺	·996,6847	·997,2585⁻	·997,7314	·998,1215⁻	·998,4435⁺	·998,7097	·998,9297
·30	·42857	·997,2018	·997,7027	·998,1126	·998,4483	·998,7235⁺	·998,9493	·999,1347
·31	·44928	·997,6426	·998,0787	·998,4329	·998,7210	·998,9555⁺	·999,1465⁺	·999,3022
·32	·47059	·998,0179	·998,3964	·998,7017	·998,9482	·999,1474	·999,3084	·999,4387
·33	·49254	·998,3869	·998,6645⁺	·998,9268[5]	·999,1371	·999,3057	·999,4410	·999,5497
·34	·51515⁺	·998,6075⁻	·998,8903	·999,1150⁺	·999,2938	·999,4360	·999,5494	·999,6398
·35	·53846	·998,8366	·999,0800	·999,2720	·999,4235⁻	·999,5432	·999,6378	·999,7127
·36	·56250[c]	·999,0303	·999,2392	·999,4026	·999,5306	·999,6310	·999,7097	·999,7715⁻
·37	·58730	·999,1937	·999,3724	·999,5111	·999,6189	·999,7027	·999,7680	·999,8188
·38	·61290	·999,3312	·999,4836	·999,6010	·999,6915⁻	·999,7613	·999,8152	·999,8568
·39	·63934	·999,4468	·999,5763	·999,6753	·999,7510	·999,8089	·999,8532	·999,8872
·40	·66667	·999,5436	·999,6534	·999,7365⁺	·999,7996	·999,8475⁻	·999,8839	·999,9115⁺
·41	·69492	·999,6245⁺	·999,7172	·999,7869	·999,8393	·999,8787	·999,9084	·999,9308
·42	·72414	·999,6920	·999,7700	·999,8282	·999,8715⁺	·999,9039	·999,9280	·999,9461
·43	·75439	·999,7481[5]	·999,8136	·999,8619	·999,8977	·999,9241	·999,9437	·999,9582
·44	·78571	·999,7947	·999,8494	·999,8895⁻	·999,9188	·999,9403	·999,9561	·999,9677
·45	·81818	·999,8332	·999,8788	·999,9118	·999,9358	·999,9532	·999,9659	·999,9751
·46	·85185⁺	·999,8650⁻	·999,9027	·999,9299	·999,9494	·999,9635⁺	·999,9736	·999,9809
·47	·88679	·999,8911	·999,9223	·999,9445⁻	·999,9603	·999,9716	·999,9797	·999,9855⁻
·48	·92308	·999,9124	·999,9381	·999,9562	·999,9690	·999,9781	·999,9845⁻	·999,9890
·49	·96078	·999,9299	·999,9509	·999,9656	·999,9759	·999,9831	·999,9881	·999,9917
·50	1·00000	·999,9441	·999,9613	·999,9731	·999,9814	·999,9871	·999,9910	·999,9937

TABLE I (*continued*).

$n=$ / $\frac{1}{2}(n-1)=$		23 / 11·0	24 / 11·5	25 / 12·0	26 / 12·5	27 / 13·0	28 / 13·5	29 / 14·0
$B(\frac{1}{2}, \frac{1}{2}(n-1))=$		·5405,2037	·5283,7848	·5170,1948	·5063,6271	·4963,3870	·4868,8722	·4779,5579
x	z^2							
·50	1·00000	·999,9441	·999,9613	·999,9731	·999,9814	·999,9871	·999,9910	·999,9937
·51	1·04081	·999,9556	·999,9696	·999,9791	·999,9856	·999,9901	·999,9932	·999,9958
·52	1·08333	·999,9649	·999,9762	·999,9838	·999,9890	·999,9925+	·999,9949	·999,9965+
·53	1·12766	·999,9724	·999,9815−	·999,9875+	·999,9916	·999,9944	·999,9962	·999,9974
·54	1·17391	·999,9784	·999,9856	·999,9905−	·999,9936	·999,9958	·999,9972	·999,9981
·55	1·22222	·999,9832	·999,9889	·999,9927	·999,9952	·999,9968	·999,9979	·999,9986
·56	1·27273	·999,9870	·999,9915+	·999,9945−	·999,9964	·999,9977	·999,9985−	·999,9990
·57	1·32558	·999,9899	·999,9935+	·999,9958	·999,9973	·999,9983	·999,9989	·999,9993
·58	1·38095+	·999,9923	·999,9951	·999,9969	·999,9980	·999,9987	·999,9992	·999,9995−
·59	1·43902	·999,9941	·999,9963	·999,9977	·999,9985+	·999,9991	·999,9994	·999,9996
·60	1·50000	·999,9956	·999,9973	·999,9983	·999,9989	·999,9993	·999,9996	·999,9997
·61	1·56410	·999,9967	·999,9980	·999,9988	·999,9992	·999,9995+	·999,9997	·999,9998
·62	1·63158	·999,9975+	·999,9985	·999,9991	·999,9995−	·999,9997	·999,9998	·999,9999
·63	1·70270	·999,9982	·999,9989	·999,9993	·999,9996	·999,9998	·999,9999	·999,9999
·64	1·77778	·999,9986	·999,9992	·999,9995+	·999,9997	·999,9998	·999,9999	1·000,0000
·65	1·85714	·999,9990	·999,9994	·999,9997	·999,9998	·999,9999	·999,9999	
·66	1·94118	·999,9993	·999,9996	·999,9998	·999,9999	·999,9999	1·000,0000	
·67	2·03030	·999,9995−	·999,9997	·999,9998	·999,9999	·999,9999		
·68	2·12500	·999,9996	·999,9998	·999,9999	·999,9999	1·000,0000		
·69	2·22581	·999,9997	·999,9999	·999,9999	1·000,0000			
·70	2·33333	·999,9998	·999,9999	·999,9999				
·71	2·44828	·999,9999	·999,9999	1·000,0000				
·72	2·57143	·999,9999	1·000,0000					
·73	2·70370	·999,9999						
·74	2·84615+	1·000,0000						
·75	3·00000							
·76	3·16667							
·77	3·34783							
·78	3·54545+							
·79	3·76190							
·80	4·00000							
·81	4·26316							
·82	4·55556							
·83	4·88235+							
·84	5·25000							
·85	5·66667							
·86	6·14286							
·87	6·69231							
·88	7·33333							
·89	8·09091							
·90	9·00000							
·91	10·11111							
·92	11·50000							
·93	13·28571							
·94	15·66667							
·95	19·00000							
·96	24·00000							
·97	32·33333							
·98	49·00000							
·99	99·00000							
1·00	∞							

TABLE I (*continued*).

$n=$ / $\frac{1}{2}(n-1)=$		30 / 14·5	31 / 15·0	Normal Curve	$n=$ / $\frac{1}{2}(n-1)=$		30 / 14·5	31 / 15·0	Normal Curve
$B(\frac{1}{2},\frac{1}{2}(n-1))=$		·4694,9840	·4614,7455+	S.D.$=\frac{1}{\sqrt{28}}$	$B(\frac{1}{2},\frac{1}{2}(n-1))=$		·4694,9840	·4614,7455+	S.D.$=\frac{1}{\sqrt{28}}$
x	z^2				x	z^2			
·00	·00000	·500,0000	·500,0000	·500,0000	·35	·53846	·999,7719	·999,8189	·999,9484
·01	·01010	·703,7584	·706,9675+	·702,5733	·36	·56250$_e$	·999,8200	·999,8582	·999,9639
·02	·02041	·776,0346	·779,9617	·775,1541	·37	·58730	·999,8584	·999,8893	·999,9750−
·03	·03093	·824,2812	·828,4366	·823,9646	·38	·61290	·999,8890	·999,9140	·999,9828
·04	·04167	·859,6477	·863,7871	·859,9564	·39	·63934	·999,9133	·999,9333	·999,9884
·05	·05263	·886,7093	·890,6955+	·887,6174	·40	·66667	·999,9325+	·999,9485+	·999,9922
·06	·06383	·907,9293	·911,6846	·909,3682	·41	·69492	·999,9477	·999,9604	·999,9948
·07	·07527	·924,8302	·928,3127	·926,7120	·42	·72414	·999,9596	·999,9697	·999,9966
·08	·08696	·938,4343	·941,6255−	·940,6649	·43	·75439	·999,9689	·999,9769	·999,9978
·09	·09890	·949,4669	·952,3633	·951,9538	·44	·78571	·999,9762	·999,9825−	·999,9986
·10	·11111	·958,4627	·961,0707	·961,1200	·45	·81818	·999,9819	·999,9867	·999,9992
·11	·12360	·965,8267	·968,1592	·968,5777	·46	·85185+	·999,9862	·999,9900	·999,9995−
·12	·13636	·971,8725+	·973,9461	·974,6504	·47	·88679	·999,9896	·999,9925+	·999,9997
·13	·14943	·976,8464	·978,6801	·979,5952	·48	·92308	·999,9922	·999,9944	·999,9998
·14	·16279	·980,9445−	·982,5582	·983,6187	·49	·96078	·999,9942	·999,9959	·999,9999
·15	·17647	·984,3242	·985,7378	·986,8879	·50	1·00000	·999,9957	·999,9970	·999,9999
·16	·19048	·987,1129	·988,3462	·989,5393	·51	1·04081	·999,9968	·999,9978	1·000,0000
·17	·20482	·989,4145−	·990,4861	·991,6847	·52	1·08333	·999,9976	·999,9984	
·18	·21951	·991,3138	·992,2414	·993,4158	·53	1·12766	·999,9983	·999,9988	
·19	·23457	·992,8807	·993,6807	·994,8083	·54	1·17391	·999,9987	·999,9992	
·20	·25000$_e$	·994,1726	·994,8601	·995,9245−	·55	1·22222	·999,9991	·999,9994	
·21	·26582	·995,2369	·995,8257	·996,8159	·56	1·27273	·999,9994	·999,9996	
·22	·28205+	·996,1130	·996,6154	·997,5248	·57	1·32558	·999,9995+	·999,9997	
·23	·29870	·996,8333	·997,2606	·998,0860	·58	1·38095+	·999,9997	·999,9998	
·24	·31579	·997,4247	·997,7870	·998,5282	·59	1·43902	·999,9998	·999,9999	
·25	·33333	·997,9097	·998,2157	·998,8749	·60	1·50000	·999,9998	·999,9999	
·26	·35135+	·998,3068	·998,5645−	·999,1452	·61	1·56410	·999,9999	·999,9999	
·27	·36986	·998,6313	·998,8476	·999,3547	·62	1·63158	·999,9999	1·000,0000	
·28	·38889	·998,8961	·999,0770	·999,5163	·63	1·70270	1·000,0000	1·000,0000	
·29	·40845+	·999,1118	·999,2626	·999,6400					
·30	·42857	·999,2871	·999,4123	·999,7340					
·31	·44928	·999,4292	·999,5329	·999,8050−					
·32	·47059	·999,5443	·999,6298	·999,8583					
·33	·49254	·999,6371	·999,7074	·999,8979					
·34	·51515+	·999,7119	·999,7694	·999,9271					
·35	·53846	·999,7719	·999,8189	·999,9484					

TABLE II. *Values of $\mathscr{F}_x(n)$ and $\delta^2\mathscr{F}_x(n)$ from $x = \cdot00$ to $\cdot10$.*

	$n=2$	δ^2	$n=3$	δ^2	$n=4$	δ^2	$n=5$	δ^2	$n=6$	δ^2	$n=7$	δ^2
·00	·318,3099	46	·500,0000	0	·636,6198	−32	·750,0000	0	·848,8264	128	·937,5000	375
·01	·318,8428	48	·500,0000	0	·635,5572	−32	·747,5000	0	·844,5886	128	·931,2688	375
·02	·319,3806	50	·500,0000	0	·634,4913	−33	·745,0000	0	·840,3636	128	·925,0750	375
·03	·319,9233	50	·500,0000	0	·633,4222	−33	·742,5000	0	·836,1515−	129	·918,9188	375
·04	·320,4711	51	·500,0000	0	·632,3498	−33	·740,0000	0	·831,9522	129	·912,8000	375
·05	·321,0240	52	·500,0000	0	·631,2741	−34	·737,5000	0	·827,7658	130	·906,7188	375
·06	·321,5821	53	·500,0000	0	·630,1950+	−34	·735,0000	0	·823,5924	130	·900,6750	375
·07	·322,1456	54	·500,0000	0	·629,1125+	−34	·732,5000	0	·819,4320	131	·894,6688	375
·08	·322,7145+	55	·500,0000	0	·628,0266	−35	·730,0000	0	·815,2847	131	·888 7000	375
·09	·323,2889	57	·500,0000	0	·626,9372	−35	·727,5000	0	·811,1505+	132	·882,7688	375
·10	·323,8690	58	·500,0000	0	·625,8443	−36	·725,0000	0	·807,0295−	132	·876,8750	375
	$n=8$	δ^2	$n=9$	δ^2	$n=10$	δ^2	$n=11$	δ^2	$n=12$	δ^2	$n=13$	δ^2
·00	1·018,5916	763	1·093,7500	1312	1·164,1047	2037	1·230,4688	2953	1·293,4497	4075−	1·353,5156	5415
·01	1·010,1415+	761	1·082,8780	1303	1·150,6250	2015+	1·214,2095−	2911	1·274,2505−	4002	1·331,2258	5299
·02	1·001,7675+	759	1·072,1362	1294	1·137,3468	1994	1·198,2413	2869	1·255,4514	3931	1·309,4659	5186
·03	·993,4694	756	1·061,5239	1284	1·124,2680	1972	1·182,5600	2828	1·237,0455−	3860	1·288,2246	5074
·04	·985,2468	753	1·051,0400	1275	1·111,3864	1951	1·167,1616	2787	1·219,0255+	3790	1·267,4907	4964
·05	·977,0995+	750	1·040,6836	1266	1·098,6998	1929	1·152,0419	2746	1·201,3846	3721	1·247,2532	4856
·06	·969,0273	747	1·030,4538	1256	1·086,2062	1908	1·137,1968	2706	1·184,1157	3653	1·227,5014	4750+
·07	·961,0297	745	1·020,3495+	1247	1·073,9033	1887	1·122,6223	2666	1·167,2122	3586	1·208,2245+	4645+
·08	·953,1067	742	1·010,3700	1237	1·061,7891	1865+	1·108,3144	2626	1·150,6672	3519	1·189,4123	4543
·09	·945,2578	739	1·000,5142	1228	1·049,8615−	1844	1·094,2690	2587	1·134,4742	3454	1·171,0543	4441
·10	·937,4828	736	·990,7812	1219	1·038,1182	1823	1·080,4824	2548	1·118,6265+	3389	1·153,1404	4342
	$n=14$	δ^2	$n=15$	δ^2	$n=16$	δ^2	$n=17$	δ^2	$n=18$	δ^2	$n=19$	δ^2
·00	1·411,0360	6986	1·446,3086	8799	1·519,5773	10866	1·571,0449	13198	1·620,8824	15808	1·669,2352	18698
·01	1·385,5134	6812	1·437,4182	8550−	1·487,1906	10521	1·535,0392	12734	1·581,1403	15182+	1·625,6438	17912
·02	1·360,6719	6642	1·409,3827	8307	1·455,8560	10186	1·500,3070	12284	1·542,9177	14606	1·583,8437	17156
·03	1·336,4947	6476	1·382,1780	8070	1·425,5400	9860	1·466,8031	11848	1·506,1557	14038	1·543,7591	16430
·04	1·312,9651	6313	1·355,7802	7839	1·396,2100	9543	1·434,4841	11426	1·470,7975−	13489	1·505,3176	15731
·05	1·290,0668	6153	1·330,1664	7613	1·367,8343	9234	1·403,3077	11017	1·436,7882	12960	1·468,4491	15060
·06	1·267,7839	5997	1·305,3138	7392	1·340,3820	8934	1·373,2330	10621	1·404,0749	12449	1·433,0868	14416
·07	1·246,1006	5843	1·281,2005−	7177	1·313,8232	8643	1·344,2204	10238	1·372,6065−	11957	1·399,1659	13797
·08	1·225,0017	5693	1·257,8049	6967	1·288,1287	8360	1·316,2317	9867	1·342,3338	11483	1·366,6248	13202
·09	1·204,4720	5546	1·235,1059	6762	1·263,2702	8085−	1·289,2296	9508	1·313,2094	11025+	1·335,4038	12631
·10	1·184,4970	5402	1·213,0832	6562	1·239,2201	7817	1·263,1782	9160	1·285,1876	10585−	1·305,4459	12083
	$n=20$	δ^2	$n=21$	δ^2	$n=22$	δ^2	$n=23$	δ^2	$n=24$	δ^2	$n=25$	δ^2
·00	1·716,2284	21886	1·761,9705+	25379	1·806,5563	29182	1·850,0690	33310	1·892,5827	37768	1·934,1631	42566
·01	1·668,6793	20891	1·710,3591	24138	1·750,7817	27658	1·790,0339	31458	1·828,1926	35541	1·865,3265−	39913
·02	1·623,2193	19938	1·661,1615+	22955−	1·697,7730	26209	1·733,1444	29704	1·767,3564	33440	1·800,4812	37421
·03	1·579,7531	19026	1·614,2594	21827	1·647,3852	24833	1·679,2253	28044	1·709,8644	31460	1·739,3779	35081
·04	1·538,1894	18152	1·569,5399	20751	1·599,4806	23525+	1·628,1107	26473	1·655,5183	29594	1·681,7828	32884
·05	1·498,4410	17316	1·526,8955−	19725+	1·553,9286	22283	1·579,6433	24988	1·604,1316	27835+	1·627,4760	30822
·06	1·460,4243	16516	1·486,2236	18747	1·510,6049	21104	1·533,6748	23583	1·555,5285+	26178	1·576,2515−	28887
·07	1·424,0591	15751	1·447,4264	17816	1·469,3916	19985+	1·490,0645+	22255−	1·509,5432	24619	1·527,9156	27072
·08	1·389,2691	15019	1·410,4108	16928	1·430,1769	18924	1·448,6798	21000	1·466,0199	23150+	1·482,2869	25370
·09	1·355,9809	14319	1·375,0880	16083	1·392,8545−	17917	1·409,3950−	19814	1·424,8115+	21768	1·439,1953	23775+
·10	1·324,1247	13650+	1·341,3735−	15278	1·357,3237	16961	1·372,0916	18693	1·385,7800	20468	1·398,4812	22280
	$n=26$	δ^2	$n=27$	δ^2	$n=28$	δ^2	$n=29$	δ^2	$n=30$	δ^2	$n=31$	δ^2
·00	1·974,8689	47711	2·014,7532	53212	2·053,8637	59075−	2·092,2437	65310	2·129,9327	71925−	2·166,9667	78923
·01	1·901,4972	44578	1·936,7603	49541	1·971,1660	54805+	2·004,7599	60375−	2·037,5836	66252	2·069,6752	72442
·02	1·832,5834	41647	1·863,7216	46120	1·893,9489	50840	1·923,3135+	55809	1·951,8597	61026	1·979,6279	66493
·03	1·767,8342	38905+	1·795,2948	42932	1·821,8157	47160	1·847,4480	51587	1·872,2384	56213	1·896,2299	61034
·04	1·706,9755+	36341	1·731,1612	39962	1·754,3985−	43744	1·776,7412	47685−	1·798,2384	51780	1·818,9353	56028
·05	1·649,7509	33943	1·671,0237	37196	1·691,3557	40576	1·710,8029	44079	1·729,4164	47701	1·747,2434	51437
·06	1·595,9207	31703	1·614,6059	34622	1·632,3706	37638	1·649,2725+	40748	1·665,3645+	43947	1·680,6952	47229
·07	1·545,2607	29609	1·561,6503	32225+	1·577,1493	34915−	1·591,8169	37672	1·605,7073	40493	1·618,8699	43372
·08	1·497,5617	27654	1·511,9172	29996	1·525,4195−	32390	1·538,1286	34832	1·550,0994	37316	1·561,3818	39838
·09	1·452,6281	25828	1·465,1836	27921	1·476,9286	30050+	1·487,9234	32210	1·498,2231	34395+	1·507,8776	36601
·10	1·410,2772	24123	1·421,2422	25992	1·431,4428	27883	1·440,9392	29799	1·449,7862	31709	1·458,0335−	33636

ON THE APPLICATION OF CONTINUED FRACTIONS TO THE EVALUATION OF CERTAIN INTEGRALS, WITH SPECIAL REFERENCE TO THE INCOMPLETE BETA-FUNCTION.

By J. H. MÜLLER, M.Sc.

(i) The subject of the present short paper was suggested by the lectures of Professor Pearson on "Laplace."

Laplace considers $\int_t^\infty e^{-t^2}dt$ or the incomplete integral of a normal curve whose area $=\sqrt{\pi}$ and standard deviation $=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$.

Let $$I_n = \int_t^\infty t^n e^{-t^2}dt$$

$$= -\frac{1}{2}\int_t^\infty t^{n-1}d(e^{-t^2})$$

$$= \frac{1}{2}\frac{e^{-t^2}}{t^{n+1}} - \frac{1}{2}(n+1)I_{n+2},$$

$$I_0 = \int_t^\infty e^{-t^2}dt$$

$$= \frac{e^{-t^2}}{2t} - \frac{e^{-t^2}}{2^2t^3} + \frac{1.3.e^{-t^2}}{2^3t^5} - \frac{1.3.5.e^{-t^2}}{2^4t^7} + \frac{1.3.5.7}{2^4}I_8$$

$$= \frac{e^{-t^2}}{2t}\left\{1 - \frac{1}{2t^2} + \frac{1.3}{(2t^2)^2} - \frac{1.3.5}{(2t^2)^3} + \ldots + \frac{(-1)^r 1.3.5\ldots(2r-1)}{(2t^2)^r} + \ldots \right.$$

$$\left. + \frac{(-1)^{r+t}1.3\ldots(2r+t-1)}{2^{r+t}e^{-t^2}}.2tI_{2(r+t)}\right\};$$

put $q=\frac{1}{2t^2}$,

$$\therefore\ I_0 = \frac{e^{-\frac{1}{2q}}}{\sqrt{\frac{1}{2q}}}\{1 - q + 1.3.q^2 - 1.3.5.q^3 + \ldots\}.$$

This is the series considered by Laplace. It is ultimately divergent, but, for large values of t or small values of q, a very accurate approximation to I_0 may be obtained by summing the terms before the point of divergence is reached.

We write $$S = 1 - q + 1.3.q^2 - 1.3.5.q^3 + \ldots.$$

Let $$\phi(t) = \frac{1}{1-t}\left(1 - \frac{q}{(1-t)^2} + \frac{1.3.q^2}{(1-t)^4} - \ldots\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)},$$

$$= y_1 + y_2 t + y_3 t^2 + \ldots + y_{x+1} t^x + y_{x+2} t^{x+1} + \ldots \quad \ldots\ldots\ldots\ldots\text{(B)}.$$

From (B) $$q\frac{d\phi}{dt} = qy_2 + 2qy_3 t + \ldots + (x+1)\, qy_{x+2} t^x + \ldots \quad \ldots\ldots\ldots\ldots\ldots\text{(C)}.$$

From (A) $$q\frac{d\phi}{dt} = \frac{q}{(1-t)^2} - \frac{1.3.q^2}{(1-t)^4} + \frac{1.3.5.q^3}{(1-t)^6} - \ldots$$

$$= 1 - (1-t)\,\phi(t)$$

$$= 1 - (1-t)\{y_1 + y_2 t + y_3 t^2 + \ldots + y_{x+1} t^x + \ldots\} \quad \ldots\ldots\ldots\ldots\text{(D)},$$

equating coefficients of t^x in (C) and (D).

$$\therefore\ q(x+1)\, y_{x+2} = -y_{x+1} + y_x \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(E)}.$$

$$\therefore\ \frac{y_x}{y_{x+1}} = 1 + q(x+1)\frac{y_{x+2}}{y_{x+1}}.$$

$$\therefore\ \frac{y_{x+1}}{y_x} = \frac{1}{1 + q(x+1)\dfrac{y_{x+2}}{y_{x+1}}}.$$

$$\therefore\ \frac{y_1}{y_0} = \frac{1}{1 + \dfrac{q}{1 + \dfrac{2q}{1 + \dfrac{3q}{1 + \ldots.}}}}$$

y_1 is the term in (A) or (B) independent of t

$$= 1 - q + 1.3.q^2 - \ldots = S.$$

In (E) put $q = 0$. $\therefore\ y_0 = y_{1\,(q=0)} = 1.$

$$\therefore\ S = \frac{1}{1 + \dfrac{q}{1 + \dfrac{2q}{1 + \ldots}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(F)}.$$

This is the continued fraction obtained by Laplace:

$$S = \frac{2t\displaystyle\int_t^\infty e^{-t^2} dt}{e^{-t^2}}$$

$$= \frac{x\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_x^\infty e^{-\frac{1}{2}x^2} dx}{\dfrac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}}$$

$$= x\,\frac{\frac{1}{2}(1-\alpha_x)}{z_x} \text{ in the usual notation,}$$

or $$\frac{\frac{1}{2}(1-\alpha_x)}{z_x} = \frac{1}{x}.S, \text{ where } q = \frac{1}{x^2}.$$

This form of the continued fraction (F) has recently been employed in the Biometric Laboratory, in checking tables of

$$\frac{\frac{1}{2}(1-\alpha_x)}{z_x}.$$

When $x=4$, $\frac{1}{x}.S = \cdot 236{,}652{,}383$ correct to 9 places is obtained by using 15 convergents of (F).

A well-known method of converting series of such a type as S into a continued fraction is the following:

Let $s_r = (-1)^{r+1}1.3\ldots(2r+1)q^{r+1} + (-1)^{r+2}1.3\ldots(2r+3)q^{r+2} + \ldots,$

$s_{r+1} = (-1)^{r+2}1.3\ldots(2r+3)q^{r+2} + (-1)^{r+3}1.3\ldots(2r+5)q^{r+3} + \ldots.$

$\therefore\ s_r - s_{r+1} = (-1)^{r+1}1.3\ldots(2r+1)q^{r+1},$

and $s_{r-1} - s_r = (-1)^r 1.3\ldots(2r-1)q^r.$

$$\therefore\ \frac{s_r - s_{r+1}}{s_{r-1} - s_r} = -(2r+1)q.$$

$$\therefore\ 1 - \frac{s_{r+1}}{s_r} = -(2r+1)q\left(\frac{s_{r-1}}{s_r} - 1\right).$$

$$\therefore\ \frac{s_r}{s_{r-1}} = \frac{(2r+1)q}{(2r+1)q - 1 + \dfrac{s_{r+1}}{s_r}}.$$

$$\therefore\ \frac{s_0}{s_{-1}} = \frac{q}{q-1+\dfrac{3q}{3q-1+\dfrac{5q}{5q-1+\dfrac{7q}{7q-1+\ldots}}}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(G).$$

$$s_0 = -q + 1.3.q^2 - \ldots = S - 1,$$

$$s_{-1} = 1 - q + 1.3.q^2 \quad \ldots = S.$$

$\therefore\ \dfrac{S-1}{S}$ is given by (G).

$$\therefore\ S = \frac{1}{1-\dfrac{q}{q-1+\dfrac{3q}{3q-1+\ldots}}}.$$

This type of continued fraction may be termed the "equivalent" fraction.

The n^{th} convergent of this continued fraction exactly reproduces n terms of the series.

(ii) Besides the Incomplete Normal Curve Integral, two other most interesting functions in statistical analysis are the incomplete Beta and Gamma functions.

Series and their equivalent fractions, analogous to (G), may easily be found for these functions.

Let $F(n) = \int_x^\infty e^{-x}x^n\,dx \qquad x > 0$

$$= e^{-x}x^n + nF(n-1)$$

$$= e^{-x}x^n\left\{1 + \frac{n}{x} + \frac{n(n-1)}{x^2} + \ldots + \frac{n(n-1)\ldots(n-r+1)}{e^{-x}x^n}\int_x^\infty e^{-x}x^r\,dx\right\}.$$

Consider $S = 1 + nt + n(n-1)t^2 + \ldots,$ where $t = \frac{1}{x}.$

This is a terminating series if n be integral, but ultimately diverges if n be fractional. If x is large compared to n a very good approximation to $\frac{F(n)}{e^{-x}x^n}$ may be obtained by summation of the terms before the point of divergence is reached.

Let $s_r = n(n-1)\dots(n-r+1)t^r + n(n-1)\dots(n-r)t^{r+1} + \dots$.

$$\therefore \frac{s_r - s_{r+1}}{s_{r-1} - s_r} = (n-r+1)t.$$

$$\therefore (n-r+1)t\frac{s_{r-1}}{s_r} = (n-r+1)t + 1 - \frac{s_{r+1}}{s_r}.$$

$$\therefore \frac{s_r}{s_{r-1}} = \frac{(n-r+1)t}{1+(n-r+1)t - \frac{s_{r+1}}{s_r}}.$$

$$s_0 = 1 + nt + n(n-1)t^2 + \dots,$$

$$s_1 = s_0 - 1.$$

$\therefore$ put $r = 1$.

$$\therefore \frac{s_0 - 1}{s_0} = \frac{nt}{1+nt-} \frac{(n-1)t}{1+(n-1)t-} \frac{(n-2)t}{1+(n-2)t-} \quad \dots\dots\dots\dots\text{(H)},$$

and

$$s_0 = \frac{1}{1-(H)}.$$

(iii) Let $B_x(u, v) = \int_0^x x^{u-1}(1-x)^{v-1}\,dx$

$$= \frac{x^u}{u}(1-x)^{v-1} + \frac{v-1}{u}\int_0^x x^u(1-x)^{v-2}\,dx$$

$$= \frac{x^u}{u}(1-x)^{v-1} + \frac{v-1}{u}B_x(u+1, v-1);$$

put $1-x = y$, $\frac{x}{y} = t$ and $I_x(u, v) = \frac{B_x(u, v)}{B_1(u, v)}$.

Hence

$$I_x(u, v) = \frac{x^u y^{v-1}}{uB_1(u, v)}\left\{1 + \frac{v-1}{u+1}t + \frac{(v-1)(v-2)}{(u+1)(u+2)}t^2 + \dots + \frac{(v-1)(v-2)\dots(v-s+1)}{(u+1)(u+2)\dots(u+s-1)}t^{s-1}\right\} + I_x(u+s, v-s).$$

If v be an integer (s) the series terminates after s terms and $I_x(u, v)$ becomes

$$\frac{x^u y^{v-1}}{uB_1(u, v)}\left\{1 + \frac{v-1}{u+1}t + \dots + \frac{(v-1)(v-2)\dots 2.1}{(u+1)(u+2)\dots(u+v-1)}t^{s-1}\right\} \quad \dots\dots\text{(I)};$$

put $k = u + v - 1$, then the first v terms of

$$(x+y)^k = x^k + kx^{k-1}y + \frac{k!}{2!\,(k-2)!}x^{k-2}y^2 + \dots + \frac{k!\,x^{k-v+2}y^{v-2}}{(v-2)!\,(k-v+2)!} + \frac{k!}{(v-1)!\,(k-v+1)!}x^{k-v+1}y^{v-1}.$$

Writing the terms backwards, the first v terms

$$= x^u y^{v-1} \frac{k!}{(v-1)!\,u!}$$
$$\times \left\{1 + \frac{v-1}{u+1}t + \frac{(v-1)(v-2)}{(u+1)(u+2)}t^2 + \ldots + \frac{(v-1)!}{(u+1)(u+2)\ldots(u+v-1)}t^{v-1}\right\}.$$

Hence if v is an integer

$$I_x(u, v) = \text{first } v \text{ terms of } (x+y)^k,$$

where $$k = u + v - 1, \quad y = 1 - x.$$

(See Karl Pearson, *Biometrika*, Vol. XVI. p. 202, and Soper in *Tracts for Computers*, No. VII.)

Let $$s' = \frac{v}{u} + \frac{v(v-1)}{u(u+1)}t + \ldots + \frac{v(v-1)\ldots(v-r)}{u(u+1)\ldots(u+r)}t^r + \ldots,$$

$$s_r = \frac{v(v-1)\ldots(v-r)}{u(u+1)\ldots(u+r)}t^r + \frac{v(v-1)\ldots(v-r-1)}{u(u+1)\ldots(u+r+1)}t^{r+1} + \ldots.$$

$$\therefore \quad s_r - s_{r+1} = \frac{v(v-1)\ldots(v-r)}{u(u+1)\ldots(u+r)}t^r,$$

$$s_{r-1} - s_r = \frac{v(v-1)\ldots(v-r+1)}{u(u+1)\ldots(u+r-1)}t^{r-1}.$$

$$\therefore \quad 1 - \frac{s_{r+1}}{s_r} = \frac{v-r}{u+r}t\left(\frac{s_{r-1}}{s_r} - 1\right),$$

or $$\frac{s_r}{s_{r-1}} = \frac{\dfrac{v-r}{u+r}t}{1 + \dfrac{v-r}{u+r}t - \dfrac{s_{r+1}}{s_r}}.$$

$$\therefore \quad \frac{s_1}{s_0} = \frac{\dfrac{v-1}{u+1}t}{1 + \dfrac{v-1}{u+1}t - \dfrac{s_2}{s_1}} = \frac{\dfrac{v-1}{u+1}t}{A} \text{ say,}$$

$$s_0 = s' \quad \text{and} \quad s_1 = s_0 - \frac{v}{u}.$$

$$\therefore \quad 1 - \frac{\dfrac{v-1}{u+1}t}{A} = \frac{\dfrac{v}{u}}{s_0}.$$

$$\therefore \frac{v}{u} + 1 - \frac{\dfrac{v-1}{u+1}t}{A} = \frac{v}{u}\left(\frac{s_0+1}{s_0}\right).$$

$$\therefore \quad \frac{s_0}{1+s_0} = \frac{\dfrac{v}{u}}{1 + \dfrac{v}{u} - 1 + \dfrac{\dfrac{v-1}{u+1}t}{1 + \dfrac{v-1}{u+1}t - 1 + \dfrac{\dfrac{v-2}{u+2}t}{1 + \dfrac{v-2}{u+2}t -}}}, \ldots \quad \ldots\ldots\ldots\ldots\text{(J).}$$

If $\dfrac{s_0}{1+s_0} = F$, then $s_0 = \dfrac{F}{1-F}$.

It is therefore seen that continued fractions (G), (H) and (I) of the "Equivalent" type are obtained for the "Normal Integral," the "Gamma Function" and the "Beta Function" series. In the Normal series, another type of continued fraction (F) was obtained by Laplace. It appeared desirable to investigate whether a similar type to (F) also exists for the Gamma and Beta Function series.

In my investigations I found that another class had actually been found for the Gamma Function (De Morgan, *Differential Calculus*, p. 590).

De Morgan finds for

$$1+\frac{n}{x}+\frac{n(n-1)}{x^2}+\frac{n(n-1)(n-2)}{x^3}+\ldots$$

the continued fraction

$$\frac{1}{1-}\ \frac{\frac{n}{x}}{1+}\ \frac{\frac{1}{x}}{1+}\ \frac{\frac{1-n}{x}}{1+}\ \frac{\frac{2}{x}}{1+}\ \frac{\frac{2-n}{x}}{1+}\ \frac{\frac{3}{x}}{1+}\ \frac{\frac{3-n}{x}}{1+}\ \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(K)}.$$

Finally I came across an important paper of Thomas Muir, "New General Formulae for the Transformation of Infinite Series into Continued Fractions," *Transactions of the Royal Society of Edinburgh*, Vol. XXVII. p. 467.

Assume $$1+B_1x+B_2x^2+\ldots=\frac{1}{a_1+}\ \frac{x}{a_2+}\ \frac{x}{a_3+}\ \ldots,$$

where $a_1, a_2, \ldots$ are independent of x. $a_1, a_2, \ldots$ are then obtained in the form of determinants in $B_1, B_2, \ldots$ by equating coefficients of like powers of x. Thus

$$1+B_1x+B_2x^2+B_3x^3+\ldots$$

$$=\frac{1}{1-}\ \frac{\beta_1x}{1-}\ \frac{\beta_2x}{\beta_1-}\ \frac{\beta_3x}{\beta_2-}\ \frac{\beta_1\beta_4x}{\beta_3-}\ \frac{\beta_2\beta_5x}{\beta_4-}\ \frac{\beta_3\beta_6x}{\beta_5-}\ \frac{\beta_1\beta_7x}{\beta_6-}\ \ldots \quad \ldots\text{(L)},$$

where the values of $\beta_1, \beta_2, \beta_3, \beta_4, \ldots$ are given by

$$B_1,\ \begin{vmatrix} 1 & B_1 \\ B_1 & B_2 \end{vmatrix},\ \begin{vmatrix} B_1 & B_2 \\ B_2 & B_3 \end{vmatrix},\ \begin{vmatrix} 1 & B_1 & B_2 \\ B_1 & B_2 & B_3 \\ B_2 & B_3 & B_4 \end{vmatrix},\ \begin{vmatrix} B_1 & B_2 & B_3 \\ B_2 & B_3 & B_4 \\ B_3 & B_4 & B_5 \end{vmatrix},$$

$$\begin{vmatrix} 1 & B_1 & B_2 & B_3 \\ B_1 & B_2 & B_3 & B_4 \\ B_2 & B_3 & B_4 & B_5 \\ B_3 & B_4 & B_5 & B_6 \end{vmatrix},\ \begin{vmatrix} B_1 & B_2 & B_3 & B_4 \\ B_2 & B_3 & B_4 & B_5 \\ B_3 & B_4 & B_5 & B_6 \\ B_4 & B_5 & B_6 & B_7 \end{vmatrix}, \ldots.$$

This method I applied to the Normal series

$$S=1-x+1.3x^2-1.3.5x^3+\ldots,$$

and found $B_1=-1$, $B_2=1.3$,

$B_3=-1.3.5$, $B_4=1.3.5.7$,

$B_5=-1.3.5.7.9$, $B_6=1.3.5.7.9.11, \ldots.$

The values found from the determinants were

$\beta_1=1$, $\beta_2=2$, $\beta_3=6$, $\beta_4=48$, $\beta_5=-45.16$, $\beta_6=45.48.16$, $\beta_7=2^8.3^4.5^2.7$.

On substituting these values and simplifying,

$$s=\frac{1}{1+}\,\frac{x}{1+}\,\frac{2x}{1+}\,\frac{3x}{1+}\,\frac{4x}{1+}\,\frac{5x}{1+}\cdots,$$

i.e. the same form as Laplace's. Compare (F).

This method was next applied to the Gamma Function series

$$S=1+nt+n(n-1)t^2+n(n-1)(n-2)t^3+\ldots.$$

Hence $B_1=n,\quad B_2=n(n-1),\quad B_3=n(n-1)(n-2),\ldots.$

The values found from the determinants were

$$\beta_1=n,\quad \beta_2=-n,\quad \beta_3=-n^2(n-1),\quad \beta_4=-2n^2(n-1),\quad \beta_5=-2n^3(n-1)^2(n-2).$$

Substituting these values, and putting $t=\frac{1}{x}$, we find

$$s=\frac{1}{1-}\,\frac{\frac{n}{x}}{1+}\,\frac{\frac{1}{x}}{1+}\,\frac{\frac{1-n}{x}}{1+}\,\frac{\frac{2}{x}}{1+}\,\frac{\frac{2-n}{x}}{1+}\cdots,$$

which is the same as De Morgan's form. Compare (K).

Tables of the Incomplete Beta Function $B_x(u,v)$ are at present being computed; such tables are of necessity limited, and it therefore seems highly important to investigate every channel which may lead to values of this function, as such may be of assistance in the actual computation of the tables, or in obtaining values outside their range.

Attempts to obtain a continued fraction of Laplace's or De Morgan's type, by employing their methods, were unsuccessful. Muir's method was then applied to the series (see (I) above)

$$S=1+\frac{v-1}{u+1}t+\frac{(v-1)(v-2)}{(u+1)(u+2)}t^2+\ldots,$$

where $$t=\frac{x}{y}=\frac{x}{1-x}.$$

Further let $$k=u+v-1,\quad u_1=\frac{v-1}{u+1},\ \ldots\ u_r=\frac{v-r}{u+r}.$$

The following values were found:

$$\beta_1=u_1,$$

$$\beta_2=-\frac{(k+1)u_1}{(u+1)(u+2)},$$

$$\beta_3=-\frac{(k+1)u_1^2u_2}{(u+2)(u+3)},$$

$$\beta_4=-\frac{2(k+1)^2(k+2)u_1^2u_2}{(u+1)(u+2)^2(u+3)^2(u+4)},$$

$$\beta_5=-\frac{2(k+1)^2(k+2)u_1^3u_2^2u_3}{(u+2)(u+3)^2(u+4)^2(u+5)},$$

$$\beta_6=\frac{2^2.3.(k+1)^3(k+2)^2(k+3)u_1^3u_2^2u_3}{(u+1)(u+2)^2(u+3)^3(u+4)^3(u+5)^2(u+6)},$$

$$\beta_7=\frac{2^2.3.(k+1)^3(k+2)^2(k+3)u_1^4u_2^3u_3^2u_4}{(u+2)(u+3)^2(u+4)^3(u+5)^3(u+6)^2(u+7)}.$$

Substituting these values in (L) we find if

$$I_x(u, v) = \frac{B_x(u, v)}{B_1(u, v)} = \frac{\int_0^x x^{u-1}(1-x)^{v-1}\,dx}{\int_0^1 x^{u-1}(1-x)^{v-1}\,dx}$$

that $$I_x(u, v) = C\left[\frac{b_1}{1+}\,\frac{b_2}{1+}\,\frac{b_3}{1+}\,\frac{b_4}{1+}\cdots\right] \quad \text{.....(M),}$$

where $$C = \frac{x^u y^{v-1}}{u \,.\, B_1(u, v)} = x^u y^{v-1}\,\frac{\Gamma(k+1)}{\Gamma(u+1)\Gamma(v)},$$

$$b_1 = 1,$$

$$b_2 = -u_1 t,$$

$$b_3 = \frac{(k+1)}{(u+1)(u+2)}\,t,$$

$$b_4 = -\frac{(u+1)(u+2)}{(u+2)(u+3)}\,u_2 t,$$

$$b_5 = \frac{2(k+2)}{(u+3)(u+4)}\,t,$$

$$b_6 = -\frac{(u+2)(u+3)}{(u+4)(u+5)}\,u_3 t,$$

$$b_7 = \frac{3(k+3)}{(u+5)(u+6)}\,t,$$

$$b_8 = -\frac{(u+3)(u+4)}{(u+6)(u+7)}\,u_4 t,$$

.....

$$b_{2r} = -\frac{(u+r-1)(u+r)}{(u+2r-2)(u+2r-1)}\,u_r t,$$

$$b_{2r+1} = \frac{r(k+r)}{(u+2r-1)(u+2r)}\,t.$$

This I believe to be a new expression for the Incomplete B-function. In computing any value $I_x(u, v)$, there is an alternative given by $I_x(u, v) = 1 - I_y(v, u)$. From general considerations it appears that the particular form to be selected should be that which does not require the binomial $(x+y)^k$ to be summed through its largest term.

The largest term of this binomial is the $(r+1)^{\text{th}}$, where r is the greatest integer consistent with

$$\frac{y(k+1)}{x+y} > r \quad \text{or} \quad y(u+v) > r, \text{ since } x+y=1.$$

With integration to the mode, there are two courses, (a) from the left of the mode, (b) from the right of the mode. Consider

$$I_x(u, v) \text{ and } I_{x'}(v, u), \quad u < v \qquad (u, v > 1).$$

In the associated frequency distribution of $I_x(u, v)$, the mode is between the origin and the mean, and the distance of the mode from the origin is

$$\frac{u-1}{u+v-2} = x < \frac{1}{2};$$

$$\therefore\ y(u+v) = \frac{(v-1)(u+v)}{u+v-2} = v - 1 + \frac{2(v-1)}{u+v-2}$$

$$= v + \text{a positive fraction.}$$

Therefore the $(v+1)^{\text{th}}$ term is the largest in the binomial.

Considering $I_{x'}(v, u)$, the mean is between the mode and the origin. If integration to the mode be attempted, the greatest term of the binomial is given by

$$u - 1 + \frac{2(u-1)}{u+v-2} > r,$$

i.e. $$u - \text{a positive fraction} > r.$$

Therefore the u^{th} term is the largest.

In computing the value of $I_x(u, v)$ from the continued fraction, it is necessary first to calculate the values of the b's; this is very simply done on a calculating machine. Since

$$(u+r)(u+r-1) = (u+r-1)(u+r-2) + 2(u+r-1),$$

the denominators of $b_3, b_4, \ldots$ can be computed continuously for a long series, and each value so obtained multiplied by $1-x$, since $t = \dfrac{x}{1-x}$; this stage completes the calculation of denominators. u_r contains a factor $u+r$, which cancels out.

The numerators will be already in part found, or can be easily calculated.

If the n^{th} convergent of the continued fraction be $\dfrac{p_n}{q_n}$,

$$p_1 = 1, \quad p_2 = 1, \quad q_1 = 1, \quad q_2 = 1 + b_2;$$

further values are given by $p_n = p_{n-1} + b_n p_{n-2}$ and $q_n = q_{n-1} + b_n q_{n-2}$.

Successive values of p_n and q_n are therefore continuously obtained. In the other or "equivalent" type of continued fraction

$$F = \frac{b_1}{1 + b_1 -} \frac{b_2}{1 + b_2 -} \frac{b_3}{1 + b_3 -}, \ldots \quad \ldots\ldots\ldots\ldots\text{compare (J).}$$

It is easily seen that $p_1 = b_1$, $q_1 = 1 + b_1$, $p_2 = b_1(1 + b_2)$, $q_2 = 1 + p_2$, and generally $q_n = 1 + p_n$.

The n^{th} convergent $$F_n = \frac{p_n}{q_n} = \frac{p_n}{1 + p_n},$$

and the series to n terms is $$\frac{F_n}{1 - F_n} = p_n,$$

where $$p_n = (1 + b_n) p_{n-1} - b_n p_{n-2}$$

$$= p_{n-1} + b_n (p_{n-1} - p_{n-2}),$$

and $$I_x(u, v) = \frac{x^u y^{v-1}}{v \,.\, B_1(u, v)} \times p_n,$$

if n terms of the series be taken to represent $I_x(u, v)$ approximately.

Illustration.

$I_{\cdot 308}(55, 91) = \cdot 0401\ 5474$ by Weddle's formula.

New Type.

$C = \cdot 0128\ 7968\ 94$; $I = C.\frac{p_r}{q_r}$.

$p_1 = b_1$; $p_2 = b_1$; $p_r = p_{r-1} + b_r p_{r-2}$.

$q_1 = 1$; $q_2 = 1 + b_2$; $q_r = q_{r-1} + b_r q_{r-2}$.

Equivalent Type.

$C = \cdot 0077\ 8442\ 764$; $I = C p_r$.

$p_1 = b_1$; $p_2 = b_1 + b_1 b_2$; $p_r = p_{r-1} + b_r \Delta p_{r-2}$

r	b_r	p_r	q_r	$I_x(u, v)$	b_r	p_r	Δp_r	$I_x(u, v)$
1	1·0	1·0	1·0	·0129	1·6545 4545 455	1·6545 4545 45	1·1835 2601 16	·0128 8
2	− ·7153 1791 908	1·0	·2846 8208 09	·0452	·7153 1791 908	2·8380 7146 61	·8225 0316 98	·0220 9
3	+ ·0203 5797 586	1·0203 5797 59	·3050 4005 68	·0431	·6949 5994 321	3·6605 7463 59	·5554 3965 31	·0284 9
4	− ·6709 9580 724	·3493 6216 86	·1140 1957 42	·0394 6	·6753 0396 651	4·2160 1428 90	·3645 4298 38	·0328 2
5	+ ·0382 3947 730	·3883 8012 43	·1256 8414 65	·0398 0	·6563 1429 411	4·5805 5727 28	·2325 6297 10	·0356 5
6	− ·6306 6522 974	·1680 4955 20	·0537 7596 55	·0402 5	·6379 5761 079	4·8131 2024 38	·1442 3620 25	·0374 7
7	+ ·0539 9412 489	·1890 1979 69	·0605 6217 10	·0401 99	·6202 0278 594	4·9573 5644 63	·0869 7741 54	·0385 9
8	− ·5938 4122 540	·0892 2504 50	·0286 2778 58	·0401 43	·6030 2069 737	5·0443 3386 17	·0510 0217 10	·0392 7
9	+ ·0679 1389 562	·1020 6211 58	·0327 4079 87	·0401 495	·5863 8407 193	5·0953 3603 27	·0290 8487 25	·0396 6
10	− ·5601 1159 281	·0520 8613 37	·0167 0604 40	·0401 563	·5702 6734 104	5·1244 2090 52	·0161 3182 30	·0398 9
11	+ ·0802 4399 733	·0602 7600 58	·0193 3329 66	·0401 554	·5546 4650 956	5·1405 5272 82	·0087 0310 30	·0400 16
12	− ·5291 2405 514	·0327 1597 96	·0104 9372 69	·0401 5462	·5394 9903 661	5·1492 5583 12	·0045 6742 08	·0400 84
13	+ ·0911 9144 164	·0382 1263 54	·0122 5675 80	·0401 5474	·5248 0372 703	5·1538 2325 20	·0023 3185 39	·0401 20
14					·5105 4063 244	5·1561 5510 59	·0011 5821 08	·0401 38
15					·4966 9096 088	5·1573 1331 67	·0005 5969 03	·0401 467
16					·4832 3699 422	5·1578 7300 70	·0002 6314 51	·0401 511
17					·4701 6201 254	5·1581 3615 21	·0001 2037 57	·0401 531
18					·4574 5022 479	5·1582 5652 78	·0000 5357 77	·0401 5407
19					·4450 8670 520	5·1583 1010 55	—	·0401 5449
20					·4331 0808 172	5·1583 3331 0	—	·0401 5467

In the Tables on pp. 293—4 I have compared the convergence of the two types of continued fractions (distinguished as the "New" and "Equivalent" Types*) to the values found by "Weddling." In one case the computation is illustrated in detail.

Other Illustrations of the Degree of Approximation of the method.

First values given by Weddling:
"*New*," "*Equiv.*" = values given by New and Equivalent Types of Continued Fractions.
(n) denotes number of convergents.

$I_{\cdot 5}(91, 55)$	·0013 3410
New (3)	·0013 7
Equiv. „	·0011 1
New (4)	·0013 30
Equiv. „	·0012 2
New (7)	·0013 3425
Equiv. „	·0013 20
New (8)	·0013 3407
Equiv. „	·0013 28
New (9)	·0013 3409
Equiv. „	·0013 312
„ (12)	·0013 339
„ (15)	·0013 3409
$I_{\cdot 5}(113, 101)$	2055 0427
New (3)	·2687
Equiv. „	·097
New (4)	·1778
Equiv. „	·126
New (7)	·2108
Equiv. „	·163
New (8)	·2028
Equiv. „	·178
New (11)	·2059
Equiv. „	·196
New (12)	·2053
Equiv. „	·1988
New (15)	·2055 3
Equiv. „	·2036
New (16)	·2055 49
Equiv. „	·2042 6
New (19)	·2055 045
Equiv. „	·2052 1
„ (25)	·2054 95
„ (30)	·2055 040
$I_{\cdot 6}(81, 21)$	·0000 1221
New (3)	·0000 1227
Equiv. „	·0000 1175
New (4)	·0000 1219
Equiv. „	·0000 1207
New (7)	·0000 1220
Equiv. „	·0000 1220

$I_{\cdot 2}(21, 81)$	·4603 9850
New (3)	·710
Equiv. „	·257
New (4)	·361
Equiv. „	·318
New (7)	·4705
Equiv. „	·4219
New (8)	·4559
Equiv. „	·4376
New (11)	·4606
Equiv. „	·4567
New (12)	·4603 1
Equiv. „	·4585
New (15)	·4604 01
Equiv. „	·4602 1
New (16)	·4603 9773
Equiv. „	·4603 2
New (17)	·4603 9834
Equiv. „	·4603 66
„ (22)	·4603 9824
$I_{\cdot 8}(81, 21)$	·5396 0150
New (3)	1·245
Equiv. „	·284
New (4)	·263
Equiv. „	·3600
New (7)	·599
Equiv. „	·497
New (8)	·494
Equiv. „	·517
New (11)	·543
Equiv. „	·538
New (12)	·537
Equiv. „	·5388 6
New (15)	·5397 2
Equiv. „	·5395 86
New (16)	·5395 0
Equiv. „	·5395 98
New (19)	·5396 038
Equiv. „	·5396 0154
New (20)	·5395 9982
Equiv. „	·5396 0154
New (21)	·5396 0129

$I_{\cdot 38}(55, 91)$	·5381 5900
New (3)	1·63
„ (4)	·243
„ (7)	·6426
„ (8)	·4765
„ (11)	·5476
„ (12)	·5325 1
„ (15)	·5387 4
„ (16)	·5378 2
„ (19)	·5381 83
„ (20)	·5381 46
„ (23)	·5381 595
„ (24)	·5381 587
$I_{\cdot 62}(91, 55)$	·4618 4100
New (3)	·98
„ (4)	·25
„ (7)	·5265
„ (8)	·4189
„ (11)	·4686
„ (12)	·4571 5
„ (15)	·4623 4
„ (16)	·4615 0
„ (19)	·4618 7
„ (20)	·4618 24
„ (23)	·4618 416
„ (24)	·4618 402
„ (26)	·4618 408
$I_{\cdot 572}(91, 55)$	·1017 1400
New (3)	·1161
„ (4)	·0966 5
„ (7)	·1022 5
„ (8)	·1014 9
„ (11)	·1017 32
„ (12)	·1017 033
„ (15)	·1017 144
„ (16)	·1017 1374
„ (18)	·1017 1398

$I_{\cdot 332}(55, 91)$	·1314 8814
New (3)	·152
„ (4)	·1244
„ (7)	·1322
„ (8)	·1312
„ (11)	·1315 1
„ (12)	·1314 81
„ (15)	·1314 788
„ (16)	·1314 8805
„ (19)	·1314 8820
$I_{\cdot 7}(81, 21)$	·0141 902
New (3)	·0146 9
„ (4)	·0140 8
„ (7)	·0141 935
„ (8)	·0141 892
„ (11)	·0141 9018
„ (12)	·0141 9015
$I_{\cdot 12}(21, 81)$	·0081 6185
New (3)	·0083
„ (4)	·0081 5
„ (7)	·0081 6204
„ (8)	·0081 6182 7
„ (9)	·0081 6184 7
„ (10)	·0081 6185 5
„ (11)	·0081 6185 41
„ (12)	·0081 6185 38

* The term "Equivalent" is used to denote that the continued fraction method applied is really equivalent to summing a series in which the function is expanded. It merely provides an orderly means of achieving this.

In the Equivalent Type, $b_1 = \frac{v}{u}$, $b_r = \frac{v-r+1}{u+r-1}t$, see (J).

The value of b_r in the New Type is given by (M).

The frequency distributions considered were given by

$$y = y_0 x^{u-1}(1-x)^{v-1},$$

where $u = 91$, $v = 55$, Mode = ·625, Mean = ·623, S.D. = ·040,
$u = 113$, $v = 101$, Mode = ·5283, Mean = ·5280, S.D. = ·034,
$u = 81$, $v = 21$, Mode = ·800, Mean = ·794, S.D. = ·040.

For $I_x(u, v)$ as the sum of the first v terms of $(x+y)^{u+v-1}$ I have taken

x	·308	·332	·38	·5	·572	·62	·5	·2	·6	·7	·8	·12
u	55	55	55	91	91	91	113	21	81	81	81	21
v	91	91	91	55	55	55	101	81	21	21	21	81
No. of largest term of the binomial	102	98	91	73 and 74	63	56	112 and 113	82	62	31	21	90
Dist. from Mode in terms of S.D.	1·5	1	0	3	1	·1	·8	0	5	2·5	0	3

It will also be noticed that the values obtained by the New Type of C.F. are in pairs less and greater than $I_x(u, v)$.

The following convergents are above the true value: 2, 3, 6, 7, 10, 11, 14, 15, 18, 19, 22, 23, etc.

The other convergents are below the true value. This is a necessary consequence of the form of the continued fraction $\frac{b_1}{1+}\frac{b_2}{1+}\frac{b_3}{1+}\ldots$ when the even b's are negative and less than unity. As we do not integrate beyond the mode of the distribution $x \equiv \frac{u-1}{u+v-2}$, it follows that the maximum value of t or $\frac{x}{y} = \frac{u-1}{v-1}$.

$$\therefore \text{ Max. value of } |b_2| = \frac{v-1}{u+1}\frac{x}{y} = \frac{(v-1)(u-1)}{(u+1)(v-1)} = \frac{u-1}{u+1} < 1.$$

$$\therefore \text{ In all cases } |b_2| < 1.$$

$$\therefore \text{ All even } |b\text{'s}| < 1.$$

It is a great advantage of the New Type that narrow limits, within which I_x lies, are soon obtained.

To illustrate the application of the New Type of continued fraction, I have chosen the more difficult examples; in several cases I have integrated the distribution as far as the mode; in one case, $I_{\cdot 33}(55, 91)$, slightly beyond the mode.

My attempts to find a value of the remainder after computing n convergents have not been successful.

Turning to the expression (L) it will be noticed that if $B_n, B_{n+1}, B_{n+2}, \ldots = 0$ the continued fraction terminates. The influence of the vanishing of $B_n, B_{n+1}, \ldots,$

is felt in the $(n+1)^{\text{th}}$ convergent and onward, i.e. in $\beta_n, \beta_{n+1}, \ldots$, but not till we reach $\beta_{2n-1}, \beta_{2n-2}, \ldots$ are these constants actually zero, and the continued fraction only terminates after $2n-1$ convergents. Hence if the series has n terms, the New Type of continued fraction terminates after $2n-1$ convergents, though the effect after the n^{th} of zero β-terms is already felt in the $(n+1)^{\text{th}}$ convergent.

The continued fraction terminating after $2n-1$ convergents will exactly reproduce the series. In taking n convergents of the continued fraction, exactly n terms of the series are reproduced, together with an approximation for the further terms, this approximation depending on the form of the terms of the series, previous to n.

In the Equivalent Type, the n^{th} convergent exactly reproduces n terms and no approximation to further terms (supposing them to exist) is obtained. If the series has only n terms, n terms of this type of C.F. exactly reproduce the series.

In terminating series, a point is thus ultimately reached after which the Equivalent Type converges more rapidly to the true value than the New Type does. Such a point may, however, be so far off that less convergents of the New Type may be necessary in order to obtain the required degree of accuracy.

In the foregoing, it has been assumed that u and v are integral. If v be not integral the expansion of $I_x(u, v)$ (by integration by parts, raising u and lowering v) ultimately diverges for $x > \frac{1}{2}$, if continued *ad infinitum*.

Another expansion, "raising u," may be obtained,

$$I_x(u, v) = \frac{\Gamma(u+v)}{\Gamma(u+1)\Gamma(v)} x^u y^v \left\{1 + \frac{u+v}{u+1} x + \frac{(u+v)(u+v+1)}{(u+1)(u+2)} x^2 + \ldots\right\},$$

which converges for all values of x in the range 0 to 1.

Suppose after a certain number of reductions by "Parts" we are left with $I_x(u+s, v-s)$, before negative indices start entering. We are summing along the decreasing direction of the expansion, hence

$$\frac{v-s}{u+s}\frac{x}{1-x} < 1, \text{ or } \frac{u+s}{u+v} > x, \text{ or } \frac{u+v}{u+s} x < 1.$$

Put $u+s = u'-1$, $v-s = v'+1$. By "raising" u' we obtain

$$\frac{\Gamma(u'+v')}{\Gamma(u')\Gamma(v'+1)} x^{u'-1} y^{v'+1} \left[1 + \frac{u'+v'}{u'} x + \frac{(u'+v')(u'+v'+1)}{u'(u'+1)} x^2 + \ldots \infty\right] = C.S' \text{ (say)},$$

$$\frac{u+v}{u+s} x < 1, \text{ i.e. } \frac{u'+v'}{u'-1} x < 1, \text{ or } \frac{u'+v'}{u'} x < 1.$$

Also $$\frac{u'+v'+w}{u'+w} = 1 + \frac{v'}{u'+w} < 1 + \frac{v'}{u'}, \text{ i.e. } < \frac{u'+v'}{u'};$$

put $$r = \frac{u'+v'}{u'} x.$$

$$\therefore\ C(1 + r + r^2 + \ldots \infty) > C.S' > C(1 + x + x^2 + \ldots \infty),$$

or $$\frac{C}{1-r} > C.S' > \frac{C}{1-x},$$

or $I_x(u+s, v-s)$ lies between $\frac{C}{1-r}$ and $\frac{C}{1-x}$ in value. It is simple to estimate from tables of $\log \Gamma(x)$ and of ordinary logarithms whether $I_x(u+s, v-s)$ may be negligible or not, within the accuracy desired for $I_x(u, v)$.

Hence if v be not an integer, it may be advisable to estimate $I_x(u+s, v-s)$, $v-s>0$, if $v = s +$ a fraction.

It will be found in most cases that if $u, v > 20$, $I_x(u+s, v-s)$ will prove to be negligible, with the proviso that we do not integrate through the mode of the distribution.

If $I_x(u+s, v-s)$ be not negligible, it is not advisable to use the New Type as the remainder has not been found.

A concluding remark may be made here, the two expansions for evaluating the Incomplete B-function due to Wishart*, while good in the neighbourhood of the mode or for the tails, do not give very satisfactory results for the range 1·5 to 3 times the standard deviation from the mode, and it is for this range that the new continued fraction appears to give good results with fairly low convergents.

* *Biometrika*, Vol. XIX. p. 29, Formula (27) for areas near mode; p. 23, Formula (25) for areas near the tail.

FURTHER NOTES ON THE χ^2 DISTRIBUTION.

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In a previous paper* we have discussed certain aspects of the (P, χ^2) Tests for Goodness of Fit; the following notes form an addition to that paper. They fall under three heads:

(1) Use of the previous sampling results to throw light on the way in which the distribution of χ_1^2 is modified when certain of the groups used in the process of fitting a theoretical distribution to the observations are combined together in calculating χ_1^2.

(2) An experimental examination of the adequacy of the χ^2 integral in a case of very small samples.

(3) The correction of an error introduced into the earlier paper in the section dealing with the comparison of two samples.

In the notation previously used it is supposed that a sample of N is classed into k groups containing frequencies $n_1, n_2, \ldots n_k$, while $m_1, m_2, \ldots m_k$ are a series of frequencies following the law

$$m_s = Nf(s;\ \alpha_1, \alpha_2, \ldots \alpha_c) \qquad (s = 1, 2, \ldots k) \quad \ldots\ldots(1),$$

whose values depend upon the c constants $\alpha_1, \alpha_2, \ldots \alpha_c$; these are determined by fitting (1) to the sample observations. Then

$$\chi_1^2 = \sum_{s=1}^{k} \frac{(n_s - m_s)^2}{m_s} \quad \ldots\ldots(2).$$

If the method of fitting be such as to make χ_1^2 a minimum, then it may be shown that upon certain conditions the sampling distribution of this quantity will follow approximately the law

$$\phi(\chi_1^2)\, d\chi_1^2 = \text{constant}\ (\chi_1^2)^{\frac{k-c-3}{2}}\, e^{-\frac{1}{2}\chi_1^2}\, d\chi_1^2 \quad \ldots\ldots(3).$$

Two of these conditions are as follows:

(a) That none of the expected frequencies m_s shall be too small.

(b) That the number of groups used in the process of fitting shall be the same as those used in calculating χ_1^2.

* *Biometrika*, Vol. xx^A. pp. 263—294, "On the Use and Interpretation of Certain Test Criteria for purposes of Statistical Inference, Part II." Dr W. F. Sheppard in a paper of about the same date published in the *Philosophical Transactions of the Royal Society*, Vol. 228 A, pp. 115—150 has also discussed very fully the validity of the law of equation (3), and the assumptions upon which it is based.

It often happens that some of the frequency groups contain very few observations—as in the tails of many curves—yet for convenience in practice we use the full number of observed groups in the fitting. For simplicity this is almost essential when using the method of moments. We are therefore placed in a dilemma. It is true that if we regard our problem as that of testing the hypothesis that the observed sample has arisen in random sampling from a population whose group proportions are actually the values $p_s = m_s/N$ determined by fitting, then no error is involved in calculating χ^2 from a reduced number of groups, say k', and entering the (P, χ^2) Tables with $n' = k'$. But if we look at the problem from the point of view of testing the adequacy of the law of equation (1) then we must decide the following point. Is less error involved by taking χ_1^2 from the full number of groups of which some are very small (neglecting condition (*a*)), and referring it to the distribution (3), i.e. entering Elderton's Tables with $n' = k - c$; or by taking a χ_1^2 from a smaller number of clubbed groups, k', of which none is too small, and entering the Tables with $n' = k' - c$ (neglecting condition (*b*))? We do not propose to enter into the general theoretical problem, but believe as is so often the case that a discussion of some experimental results will throw light on the issues involved.

(1) *The Effect of combining Groups.*

In section (4) of our previous paper a sampling experiment was described in which the population law followed a cubic ($c = 3$), and the area under the curve was broken into 8 groups ($k = 8$). Random samples of 200 were drawn, the expected frequencies being

Group	1	2	3	4	5	6	7	8	Total
	10·4	12·8	17·0	22·2	27·8	33·0	37·2	39·6	200·0

A cubic was fitted by the method of moments to each sample, the frequencies $m_1, m_2, \ldots m_8$ obtained, and the resulting distribution of χ_1^2 found. Within the errors of sampling this agreed satisfactorily with the distribution of equation (3), putting $k = 8$, $c = 3$. Suppose now that we use the same values of m obtained by fitting to 8 groups, but in calculating χ_1^2 club the groups together as follows:

Case (*a*); $k' = 7$, groups, $m_1 + m_2$, m_3, m_4, m_5, m_6, m_7, m_8.

Case (*b*); $k' = 6$, groups, $m_1 + m_2$, $m_3 + m_4$, m_5, m_6, m_7, m_8.

Case (*c*); $k' = 5$, groups, $m_1 + m_2$, $m_3 + m_4$, $m_5 + m_6$, m_7, m_8.

The resulting χ_1^2 distributions are shown in Table I, together with the theoretical distributions which would hold if we might use equation (3) with $k = k'$, $c = 3$. First consider the mean values of χ_1^2; if the theory were adequate we should have

Case (*a*). Mean $\chi_1^2 = k' - c - 1 = 3{\cdot}000$; standard error for 208 samples = 0·170.
Observed mean = 3·289.

Case (*b*). Mean $\chi_1^2 = k' - c - 1 = 2{\cdot}000$ " " " " " = 0·139.
Observed mean = 2·311.

Case (*c*). Mean $\chi_1^2 = k' - c - 1 = 1{\cdot}000$ " " " " " = 0·098.
Observed mean = 1·471.

The observed values are significantly and increasingly too great. If we examine Table I it is seen that this shows itself most clearly in a shortage of very small values of χ_1^2. At the other end of the distributions where a knowledge of the form of the curve is the more important in practical testing, the disagreement is not so marked. If we level up the most serious discrepancy by combining the first two groups and test for goodness of fit with the groups indicated by the bracketings, the values of P given at the bottom of the Table are obtained.

TABLE I. *Sampling Experiment; Effect of combining certain Groups.*

	7 Groups		6 Groups		5 Groups	
χ_1^2	Observation	Theory	Observation	Theory	Observation	Theory
0—1	19 } 77	41·3 } 88·9	51 } 119·5	81·8 } 131·4	105·5 } 163	142·0 } 175·3
1—2	58 }	47·6 }	68·5 }	49·6 }	57·5 }	33·3 }
2—3	38	37·6	36·5	30·1	22	15·4
3—4	30	27·1	23	18·3	10	7·8
4—5	21	18·7	7	11·1	4	4·2
5—6	16	12·5	5 } 13	6·7 } 10·8	2 } 13	2·3 } 9·5
6—7	8 } 15	8·3 } 13·7	8 }	4·1 }	2 }	1·3 }
7—8	7 }	5·4 }	5 } 9	2·5 } 6·3	5 }	1·7* }
8—9	5 }	3·5 }	1 }	1·5 }		
9—10	3 }	2·2 }	3 }	2·3* }		
10—11	2 } 11	1·4 } 9·5				
11—12	0 }	·9 }				
12—13	1 }	1·5* }				
Goodness of Fit	χ^2	3·52		6·77		5·60
	n'	7		6		4
	P	·740		·241		·136

These discrepancies are of course of the type we should expect to meet. An essential condition for entering the χ^2 Tables with $n' = k - c$ is that the χ_1^2 used shall be approximately the minimum value that can be obtained in fitting a distribution of the form of equation (1) to the observed frequencies classed into k groups. The observed values of χ_1^2 shown in Table I may now differ considerably from minimum values owing to our reducing the number of groups *after* the process of fitting. It is clear that some danger is involved in this procedure.

In the paper referred to above Sheppard has specially mentioned this point†, and suggested that the value of the constants found from the original tabulation may be used as first approximations in obtaining the values corresponding to the reduced grouping system. This is the ideal procedure, but it can rarely be followed in the course of ordinary work where a rough appreciation of the adequacy of the fit is all that is required. It should in fact be remembered that the groups we have combined in the illustration contain a large portion of the total frequency; in practice it is only the small tail groups of a fitted frequency curve that are usually

* These frequencies correspond to the remaining tail area of the theoretical curves.

† *Loc. cit.* p. 144.

clubbed together, and the χ_1^2 calculated from the resulting k' groups may still remain nearly the minimum χ_1^2 for the new system of grouping. If so, no serious error is involved in entering the Tables for Goodness of Fit with this χ_1^2 and $n' = k' - c$. In any case we may be certain on one point—that if this process shows a reasonable fit we may be content, since the true minimum χ_1^2 would show a better fit still.

(2) *The Case of very small Samples.*

The manner in which the χ^2 integral fails when the group frequencies become very small is a problem not yet fully explored. Each worker has no doubt his own lower limit—10, 8, 5 ?—for the size he will allow an expected frequency group, but he is probably not very clear why he has chosen that limit or what errors will be involved if it be exceeded. The following simple example is perhaps therefore of interest.

Suppose that repeated samples of 10 be taken from a population divided into 3 groups in proportions $p_1 = 0{\cdot}2$, $p_2 = 0{\cdot}5$, $p_3 = 0{\cdot}3$. The expected frequencies will be $\tilde{m}_1 = 2$, $\tilde{m}_2 = 5$, $\tilde{m}_3 = 3$. There will be 66 possible types of sample n_1, n_2, n_3, and for each of these it is easy to calculate

(*a*) The chance of occurrence, or

$$C = \frac{N!}{n_1!\, n_2!\, n_3!} (p_1)^{n_1} (p_2)^{n_2} (p_3)^{n_3} \quad \text{.....(4).}$$

(*b*) The value of χ^2, or

$$\chi^2 = \frac{(n_1 - \tilde{m}_1)^2}{\tilde{m}_1} + \frac{(n_2 - \tilde{m}_2)^2}{\tilde{m}_2} + \frac{(n_3 - \tilde{m}_3)^2}{\tilde{m}_3} \quad \text{.....(5).}$$

(*c*) The likelihood as previously defined, or

$$\lambda = \left(\frac{\tilde{m}_1}{n_1}\right)^{n_1} \left(\frac{\tilde{m}_2}{n_2}\right)^{n_2} \left(\frac{\tilde{m}_3}{n_3}\right)^{n_3} \quad \text{.....(6).}$$

These possible samples may be represented by 66 discrete points in a two-dimensioned space, with each of which is associated a value of C. When dealing with large samples these points increase in number and become so closely packed, that we can represent them and their associated C's by a continuous density field,

$$D = D_0 e^{-\frac{1}{2}\chi^2} \quad \text{.....(7).}$$

In this field the contours of χ^2 correspond closely to the levels both of constant C and of constant λ. But we may ask how far in the case of samples of 10 is the χ^2 integral of any value ? The position is indicated in Table II. The 66 types of sample have been arranged in descending order of C, and for each we give :

(*a*) P_c, the sum of the values of C *lower* than that associated with the sample, or the chance of drawing a less probable sample.

(*b*) P_{χ^2}, which for three groups is $e^{-\frac{1}{2}\chi^2}$, or the value of the χ^2 probability integral that would ordinarily be obtained from the Tables in the case of larger samples.

TABLE II. *Measures of Probability in very small Samples.*

n_1	n_2	n_3	P_e	P_{χ^2}	P_λ	n_1	n_2	n_3	P_e	P_{χ^2}	P_λ
3	5	2	·915	1·000	·915	7	2	1	·027	·022	·053
2	6	2	·844	·766	·724	0	9	1	·023	·035	·006
3	6	1	·844	·705	·653	7	3	0	·020	·017	·012
4	4	2	·709	·766	·795	0	5	5	·018	·024	·010
4	5	1	·709	·659	·532	6	1	3	·015	·035 ×	·034
3	4	3	·589	·705	·858	5	1	4	·015	·038 ×	·043
2	5	3	·589	·659	·596	1	3	6	·011	·006	·025
2	7	1	·482	·442	·448	2	2	6	·0090	·0066	·0266
5	4	1	·443	·362	·409	4	1	5	·0073	·0180 ×	·0229
4	3	3	·409	·442	·498	7	1	2	·0058	·0140 ×	·0155
1	7	2	·376	·344	·262	0	10	0	·0048	·0067	·0013
1	6	3	·344	·362	·319	0	4	6	·0039	·0037	·0036
5	3	2	·313	·344	·379	8	2	0	·0032	·0023	·0025
2	4	4	·285	·282	·350	3	1	6	·0025	·0037	·0050
4	6	0	·258	·282	·149	8	1	1	·0019	·0024	·0044
3	7	0	·233	·247	·114	1	2	7	·0015	·0004	·0033
3	3	4	·211	·247	·296	6	0	4	·0013	·0067	·0011
1	8	1	·189	·163	·195	7	0	3	·0011	·0044	·0007
5	5	0	·170	·189	·095	2	1	7	·0008 8	·0003 3	·0023 1
1	5	4	·151	·189	·244	5	0	5	·0006 8	·0044 4	·0008 9
2	8	0	·136	·127	·067	0	3	7	·0004 9	·0002 9	·0004 9
6	3	1	·120	·116	·228	8	0	2	·0003 7	·0012 7	·0002 6
5	2	3	·108	·163 ×	·216	4	0	6	·0002 6	·0012 7	·0003 8
4	2	4	·098	·127	·185	9	1	0	·0001 6	·0001 8	·0001 6
6	4	0	·088	·074	·057	3	0	7	·0001 2	·0001 6	·0000 9
6	2	2	·079	·091	·140	9	0	1	·0000 84	·0001 58	·0000 49
2	3	5	·070	·060	·176	1	1	8	·0000 49	·0000 13	·0001 30
1	4	5	·062	·049	·088	0	2	8	·0000 20	·0000 11	·0000 20
0	7	3	·055	·116 ×	·046	2	0	8	·0000 10	·0000 09	·0000 10
0	8	2	·048	·091 ×	·036	10	0	0	·0000 04	·0000 09	·0000 02
1	9	0	·042	·038	·017	0	1	9	·0000 02	·0000 002	·0000 07
3	2	5	·037	·043	·082	1	0	9	·0000 001	·0000 002	·0000 001
0	6	4	·031	·074 ×	·028	0	0	10	nil	·0000 0000 2	nil

(*c*) P_λ, the chance of obtaining a sample with a value of λ lower than that observed.

It will be seen that P_{χ^2} is on the whole a better approximation to P_e than to P_λ. In the most important region for tests of significance, namely between $P = \cdot10$ and $\cdot01$, a × indicates the cases of worst agreement between P_e and P_{χ^2}, but throughout the whole range the *order* of the values of the three P's can hardly be said to differ. Whether or no the χ^2 approximation will be considered here as satisfactory depends upon the degree of expectation entertained by the reader and the faith he has already placed in the test when dealing with very small samples, but the present authors must confess themselves pleasantly surprised to find so close an agreement in this rather extreme case.

(3) *The Value of Minimum χ_1^2 in the Case of two Samples.*

It is necessary to correct an error which was introduced into section (6) of our earlier paper. The problem was that of testing the hypothesis that two samples:

the first of size N with group frequencies $n_1, n_2, \ldots n_t$,
the second of size N' „ „ „ $n_1', n_2', \ldots n_t'$,

TABLE III. *Two Sample Test: Comparison of values of χ_1^2 and P.*

TABLE III. *Two Sample Test; Comparison of values of* χ_1^2 *and P.*

Example	*I (a)*		*I (b)*		*I (c)*		*I (d)*		*II (a)*		*II (b)*		*II (c)*	
	n_s	n_s'	n_s	n_s'	n_s	n_s'	n_s	n_s'	n_s	n_s'	n_s	n_s'	n_s	n_s'
Frequencies	24	52	24	58	24	64	24	67	10	12	10	15	10	17
	43	78	43	67	43	56	43	50	18	17	18	24	18	25
	33	70	33	75	33	80	33	83	22	21	22	11	22	8
Totals	100	200	100	200	100	200	100	200	50	50	50	50	50	50
χ_1^2 of (13)	·4473		2·6238		6·8473		10·1414		·2334		5·4864		9·3375	
P	·7996		·2693		·0326		·0063		·8899		·0644		·0094	
χ_1^2 of (14)	·4474		2·6257		6·8673		10·1969		·2336		5·5238		9·4877	
P	·7996		·2691		·0323		·0061		·8897		·0632		·0087	

Example	*III (a)*		*III (b)*		*III (c)*	
	n_s	n_s'	n_s	n_s'	n_s	n_s'
Frequencies	12	18	12	21	12	24
	15	16	15	17	15	17
	21	15	21	13	21	11
	46	55	46	59	46	65
	83	76	83	73	83	67
	51	60	51	63	51	65
	22	25	22	26	22	28
	20	12	20	9	20	7
	16	13	16	11	16	9
	14	10	14	8	14	7
Totals	300	300	300	300	300	300
χ_1^2 of (13)	7·2047		14·8771		24·7823	
P	·6159		·0944		·0032	
χ_1^2 of (14)	7·2406		15·0436		25·1712	
P	·6121		·0898		·0028	

have been drawn from the same population, and we considered the deduction of the usual test from the point of view of the method of likelihood. In the notation employed, Ω is the set of all possible pairs of populations with group proportions p_s and p_s' $(s = 1, 2, \ldots t)$, while ω is the subset of Ω in which the pairs are identical, or $p_s = p_s' = q_s$ $(s = 1, 2, \ldots t)$. Then the likelihood that the samples have come from some particular member of ω is

$$\lambda = \frac{C_q}{C(\Omega_{\max.})} = \prod_{s=1}^{t} \left(\frac{Nq_s}{n_s}\right)^{n_s} \left(\frac{N'q_s}{n_s'}\right)^{n_s'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

The expression (8) takes a maximum value when

$$q_s = (n_s + n_s')/(N + N') = Q_s \quad (s = 1, 2, \ldots t)\ldots\ldots(9).$$

If the sample group frequencies, however, are not too small, then the λ of (8) becomes approximately

$$\lambda = e^{-\frac{1}{2}\chi_1^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

where

$$\chi_1^2 = \sum_{s=1}^{t} \left\{\frac{(n_s - Nq_s)^2}{Nq_s} + \frac{(n_s' - N'q_s)^2}{N'q_s}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(11).$$

The values of q_s which make χ_1^2 a minimum, and therefore the λ of (10) a maximum, are not as we stated by an oversight those of (9), but may be easily shown to be

$$q_s = \left(\sqrt{\frac{n_s^2}{N} + \frac{n_s'^2}{N'}}\right) \Big/ \left(\sum_{s=1}^{t} \sqrt{\frac{n_s^2}{N} + \frac{n_s'^2}{N'}}\right)$$

$$= Q_s \sqrt{1 + \frac{x_s^2}{NN'Q_s^2}} \Big/ \left\{\sum_{s=1}^{t} \left(Q_s \sqrt{1 + \frac{x_s^2}{NN'Q_s^2}}\right)\right\} = Q_s' \quad \ldots\ldots\ldots(12),$$

where

$$n_s = NQ_s + x_s, \quad n_s' = N'Q_s - x_s.$$

These values lead to

$$\text{Minimum } \chi_1^2 = \left\{\sum_{s=1}^{t} \sqrt{\frac{n_s^2}{N} + \frac{n_s'^2}{N'}}\right\}^2 - N - N' \ldots\ldots\ldots\ldots(13).$$

If the sample group frequencies be not too small, x_s^2 will be small compared with $NN'Q_s^2$, at any rate for deviations lying within the region of significant frequency. It follows that Q_s (9), and Q_s' (12), will not differ greatly, and the true minimum χ_1^2 of (13) will be almost the same as that ordinarily used in applying this test, namely

$$\chi_1^2 = \sum_{s=1}^{t} \left\{NN' \left(\frac{n_s}{N} - \frac{n_s'}{N'}\right)^2 \Big/ (n_s + n_s')\right\} \quad \ldots\ldots\ldots\ldots\ldots(14),$$

which is obtained by taking $q_s = Q_s = (n_s + n_s')/(N + N')$, the value really maximising the λ of (8) not that of (10). The difference is of the order of approximation necessarily involved in any use of the χ^2 test. The numerical examples given in Table III, p. 303, illustrate this point; the difference between the two χ_1^2's is greatest when there are many groups, but it is clear that no error of importance would arise by using one value of P rather than the other.

Summary.

There are several ways by which to approach and to interpret the χ^2 Tests for Goodness of Fit; in all cases the use of the final integral can be considered only as an approximation. In our previous paper we discussed the use of the method of likelihood and emphasised the difference between testing a simple and testing a composite hypothesis. In the first case we obtain an answer to the question, "could this sample have come from a certain exactly specified population?" In the second, to a somewhat different question, "could it have come from a population whose distribution follows a law of a certain type depending on several undetermined parameters?" In the latter case the scheme of the test does not allow for the frequency groups being clubbed together *after* the process of fitting has been carried out. We have illustrated the effect of this clubbing on the distribution of χ^2 on the data of our previous sampling experiment. In general it would not be easy to gauge numerically the extent of the error involved, but it will probably not be large if, as is customary, the groups which are combined contain only a small portion of the total frequency.

The point at which the χ^2 distribution becomes inadequate to express the sampling variation when dealing with very small frequency groups has not yet been fully investigated. The result of an examination of the position in the case of a sample of 10 drawn from a population divided into 3 groups, suggests that the approximation is much more satisfactory than might have been expected.

In a final section we have corrected an earlier misstatement, in connection with the test for comparing two samples, and shown in a few numerical examples how the difference between minimum χ^2 and the χ^2 of maximum likelihood is not of real significance—is in fact of the general order of the χ^2 approximations.

TABLE OF THE VALUES OF THE DIFFERENCES OF THE POWERS OF ZERO.

By ETHEL M. ELDERTON, assisted by MARGARET MOUL.

At the time when the first quarter of this table was originally worked (for a special problem during the War) the authors* of it were unaware of Cayley's paper†. Later being informed of it, they checked the original work by reduction from Cayley's numbers, he tabling $\Delta^m 0^n/\Gamma(m+1)$, while they had tabled

$$\Delta^p 0^{p+s}/\Gamma(p+s+1).$$

Of the 100 entries six were found in error in the twelfth decimal place, two differed by one unit in that place, three differed by two units and one—the last entry in the table—by 5‡. Cayley takes in our notation $\Delta^p 0^{p+s}/\Gamma(p+1)$ from $p=1$ to 20 and $p+s$ from 1 to 20. Hence by reduction from Cayley's table E. M. Elderton and M. Moul were able to add half as much again to the original table in *Biometrika*, i.e. to take $p=11$ to 20 as long as $p+s$ did not exceed 20. Then, using Cayley's method, they extended his table and reduced from this extension a complete table for $p=1$ to 20 and $s=1$ to 20. Thus the table now includes all values of

$$\Delta^m 0^n/\Gamma(m+1)$$

from $m=0$ to 20 and $n=0$ to 40, and is probably the most extensive table of the differences of the powers of zero yet published.

Laplace has indicated the importance of the differences of the powers of zero in the theory of probability for problems allied to that of De Moivre, and further illustrations will be given in the forthcoming Part II of the *Book of Tables for Statisticians and Biometricians.*

* K. P. and E. M. E., *Biometrika*, Vol. xvii. p. 200.

† *Trans. Camb. Phil. Soc.* Vol. xiii. Part I (1881), pp. 1—4.

‡ 8·826,386,039,207 instead of the correct value 8·826,386,039,212.

Values of the Differences of the Powers of Zero.

Table of $q(p, s) = \dfrac{\Delta^p 0^{p+s}}{\Gamma(p+s+1)}$ *from* $p = 1$ *to* 20 *and* $s = 0$ *to* 20.

Values of p.

s	1	2	3	4	5
0	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000
1	·500,000,000,000	1·000,000,000,000	1·500,000,000,000	2·000,000,000,000	2·500,000,000,000
2	·166,666,666,667	·583,333,333,333	1·250,000,000,000	2·166,666,666,667	3·333,333,333,333
3	·041,666,666,667	·250,000,000,000	·750,000,000,000	1·666,666,666,667	3·125,000,000,000
4	·008,333,333,333	·086,111,111,111	·358,333,333,333	1·012,500,000,000	2·298,611,111,111
5	·001,388,888,889	·025,000,000,000	·143,750,000,000	·513,888,888,889	1·406,250,000,000
6	·000,198,412,698	·006,299,603,175	·050,016,534,392	·225,562,169,312	·741,732,804,233
7	·000,024 801,587	·001,405,423,280	·015,426,587,302	·087,632,275,132	·345,568,783,069
8	·000,002,755,732	·000,281,635,802	·004,284,060,847	·030,638,778,660	·144,695,216,049
9	·000,000,275,573	·000,051,256,614	·001,083,829,365	·009,760,802,469	·055,162,863,757
10	·000,000,025,052	·000,008,546,944	·000,252,086,841	·002,860,825,517	·019,341,229,758

s	6	7	8	9	10
0	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000
1	3·000,000,000,000	3·500,000,000,000	4·000,000,000,000	4·500,000,000,000	5·000,000,000,000
2	4·750,000,000,000	6·416,666,666,667	8·333,333,333,333	10·500,000,000,000	12·916,666,666,667
3	5·250,000,000,000	8·166,666,666,667	12·000,000,000,000	16·875,000,000,000	22·916,666,666,667
4	4·529,166,666,667	8·079,166,666,667	13·386,111,111,111	20·950,000,000,000	31·333,333,333,333
5	3·237,500,000,000	6·601,388,888,889	12·300,000,000,000	21·375,000,000,000	35·138,888,888,889
6	1·989,616,402,116	4·625,925,925,926	9·671,957,671,958	18·628,174,603,175−	33·604,414,682,540
7	1·077,777,777,778	2·851,851,851,852	6·679,365,079,365+	14·235,491,071,429	28·141,121,031,746
8	·523,916,997,354	1·575,358,796,296	4·127,361,937,831	9·721,510,416,667	21·034,795,249,118
9	·231,631,944,444	·790,558,449,074	2·314,315,476,190	6·017,912,946,429	14·238,267,471,340
10	·094,114,940,326	·364,277,277,988	1·190,485,668,524	3·414,504,607,083	8·826,386,039,212 *

Values of s.

s	1	2	3	4	5
11	·000,000,002,088	·000,001,315,236	·000,054,300,445−	·000,777,366,923	·006,287,061,463
12	·000,000,000,161	·000,000,187,914	·000,010,897,672	·000,197,066,149	·001,907,096,356
13	·000,000,000,011	·000,000,025,057	·000,002,048,012	·000,046,850,391	·000,542,762,985+
14	·000,000,000,001	·000,000,003,132	·000,000,361,967	·000,010,491,635−	·000,145,593,321
15	·000,000,000,000	·000,000,000,368	·000,000,060,389	·000,002,221,479	·000,036,953,700
16	·000,000,000,000	·000,000,000,041	·000,000,009,542	·000,000,446,204	·000,008,904,739
17	·000,000,000,000	·000,000,000,004	·000,000,001,432	·000,000,085,264	·000,002,043,183
18	·000,000,000,000	·000,000,000,000	·000,000,000,205−	·000,000,015,540	·000,000,447,548
19	·000,000,000,000	·000,000,000,000	·000,000,000,028	·000,000,002,707	·000,000,093,803
20	·000,000,000,000	·000,000,000,000	·000,000,000,000	·000,000,000,452	·000,000,018,851

s	6	7	8	9	10
11	·035,436,000,631	·155,444,052,796	·566,707,251,082	1·791,545,336,174	5·056,157,797,803
12	·012,447,698,996	·061,854,855,924	·251,424,842,802	·875,558,648,133	2·696,234,748,153
13	·004,102,251,152	·023,084,987,476	·104,575,173,440	·400,963,836,098	1·346,608,080,109
14	·001,274,353,342	·008,119,780,273	·040,979,983,168	·172,934,537,974	·633,142,757,534
15	·000,374,659,155−	·002,702,776,182	·015,194,003,252	·070,548,202,960	·281,476,384,198
16	·000,104,608,335−	·000,854,421,374	·005,349,474,876	·027,323,164,021	·118,769,057,007
17	·000,027,822,135−	·000,257,321,024	·001,794,174,688	·010,079,078,784	·047,721,531,774
18	·000,007,067,421	·000,074,028,765−	·000,574,831,831	·003,551,303,538	·018,311,726,897
19	·000,001,718,694	·000,020,393,546	·000,176,363,075−	·001,198,178,554	·006,727,553,604
20	·000,000,400,972	·000,005,391,171	·000,051,929,785−	·000,387,964,657	·002,371,839,420

* The table thus far was computed by K. P. and E. M. E. from the formula

$$q(p, s) = \frac{p}{p+s}\{q(p, s-1) + q(p-1, s)\}$$

and checked by the formula

$$\Delta^p 0^{p+10} = p^{p+10} - p(p-1)^{p+10} + \frac{p(p-1)}{2!}(p-2)^{p+10} - \ldots + (-1)^r \frac{p!}{(p-r)!\,r!}(p-r)^{p+10} + \ldots.$$

It was calculated for a special investigation, but it was thought that it might be of value to other computers, and was accordingly published in *Biometrika*.

Table of the Differences of the Powers of Zero (continued).

Values of p.

s	11	12	13	14	15
0	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000
1	5·500,000,000,000	6·000,000,000,000	6·500,000,000,000	7·000,000,000,000	7·500,000,000,000
2	15·583,333,333,333	18·500,000,000,000	21·666,666,666,667	25·083,333,333,333	28·750,000,000,000
3	30·250,000,000,000	39·000,000,000,000	49·291,666,666,667	61·250,000,000,000	75·000,000,000,000
4	45·161,111,111,111	63·120,833,333,333	85·962,500,000,000	114·498,611,111,111	149·604,166,666,667
5	55·206,250,000,000	83·525,000,000,000	122·407,638,888,889	174·562,500,000,000	243·125,000,000,000
6	57·465,724,206,349	93·993,816,137,566	148·064,153,439,153	225·838,657,407,407	334·974,041,005,291
7	52·315,294,312,169	92·405,753 968,254	156·305,439,814,815⁻	254·762,731,481,481	402·093,253,968,254
8	42·465,841,324,956	80·922,957,175,926	146·855,674,327,601	255·575,349,151,235⁻	428·914,306,382,275⁺
9	31·187,259,837,963	64·062,981,150,794	124·633,750,964,506	231·431,626,157,407	412·716,207,837,302
10	20·959,528,792,806	46·375,914,514,691	96·657,637,009,981	191·385,403,514,310	362·460,966,810,967
11	13·007,843,295,304	30·982,830,161,736	69·138,586,384,680	145·893,434,343,434	293·281,385,281,385⁺
12	7·510,646,020,784	19·246,738,091,260	45·960,368,727,489	103·305,893,961,266	220·326,266,245,918
13	4·059,574,796,242	11·187,030,186,001	28·573,699,456,745⁺	68·382,011,401,932	154·665,148,739,919
14	2·064,795,723,662	6·116,227,342,921	16·702,557,347,988	42·542,284,374,960	102·003,844,714,593
15	·992,653,584,094	3·159,502,634,229	9·221,670,706,029	24·989,495,556,339	63·496,670,135,466
16	·452,801,816,745⁻	1·548,130,478,989	4·827,841,910,525⁺	13·914,757,484,537	37·457,142,396,776
17	·196,634,172,633	·721,971,579,981	2·404,919,179,220	7·370,176,557,826	21·012,805,759,969
18	·081,531,203,270	·321,401,113,301	1·143,295,606,541	3·724,644,071,910	11·244,295,378,127
19	·032,361,544,187	·136,940,383,544	·520,095,870,972	1·800,798,763,647	5·755,188,591,959
20	·012,324,749,022	·055,974,424,712	·226,936,783,148	·834,949,931,033	2·824,345,081,282

Values of s.

s	16	17	18	19	20
0	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000	1·000,000,000,000
1	8·000,000,000,000	8·500,000,000,000	9·000,000,000,000	9·500,000,000,000	10·000,000,000,000
2	32·666,666,666,667	36·833,333,333,333	41·250,000,000,000	45·916,666,666,667	50·833,333,333,333
3	90·666,666,666,667	108·375,000,000,000	128·250,000,000,000	150·416,666,666,667	175·000,000,000,000
4	192·216,666,666,667	243·336,111,111,111	304·025,000,000,000	375·408,333,333,333	458·673,611,111,111
5	331·688,888,888,889	444·337,500,000,000	585·675,000,000,000	760·857,638,888,889	975·625,000,000,000
6	484·845,767,195,767	686·787,632,275,132	954·346,974,206,349	1303·555,505,952,381	1753·215,773,809,524
7	617·001,058,201,058	923·516,989,087,302	1352·062,053,571,429	1940·643,601,190,476	2736·192,129,629,630
8	697·276,909,722,222	1102·139,851,190,476	1699·062,857,142,857	2561·274,915,123,457	3783·905,031,966,490
9	710·395,595,238,095⁺	1185·119,330,357,143	1922·788,125,000,000	3042·757,062,940,917	4708·042,824,074,074
10	660·219,422,799,423	1161·879,955,691,171	1983·000,909,015,753	3292·737,981,626,784	5333·852,370,467,239
11	565·037,515,899,738	1048·485,607,751,623	1881·612,320,752,165⁻	3277·088,524,840,001	5555·446,706,642,832
12	448·779,304,083,232	877·707,017,282,501	1655·591,602,820,799	3023·255,562,114,684	5361·688,917,977,822
13	332·934,870,523,118	686·030,403,089,851	1359·651,487,302,958	2602·351,060,591,725⁺	4826·690,896,102,756
14	231·967,314,793,446	503·418,103,355,356	1047·976,644,745,302	2101·703,830,345,561	4075·526,309,675,481
15	152·497,540,608,471	348·455,185,855,783	761·690,089,418,773	1600·131,896,338,893	3243·233,260,579,640
16	94·977,341,502,623	228·434,938,336,149	524·183,838,223,194	1153·199,970,190,847	2442·462,905,982,605⁻
17	56·237,647,157,621	142·336,292,746,885⁻	342·781,781,641,755⁻	789·545,924,578,318	1747·031,800,303,742
18	31·756,208,252,117	84·559,214,770,943	213·670,498,206,349	515·165,190,078,613	1190·629,994,938,081
19	17·148,067,128,720	48·028,438,674,841	127·312,996,320,579	321·239,093,199,596	775·317,481,096,245⁻
20	8·876,627,648,890	26·145,571,013,606	72·690,900,316,193	191·914,612,225,641	483·616,046,660,943

ON THE RELATION OF THE DURATION OF PREGNANCY TO SIZE OF LITTER AND OTHER CHARACTERS IN BITCHES*.

By MARGARET AND KARL PEARSON.

(1) (i) THE following data relate to the duration of pregnancy, the age of the bitch, the size of litter, the order of the pregnancy, etc. in small dogs bred in the Biometric Laboratory, partly pure Pekinese and partly hybrids from the cross Pekinese × Pomeranian. The material is more sparse than we had hoped for, since about half the whole series of dogs were bred by Dr Usher in Scotland, and we found on examining the Scottish schedules that most of the dates of mating had not been entered, only the dates of littering; the mating books themselves had disappeared during Dr Usher's absence on war-service in Greece. It was therefore only possible to use for this enquiry the data for dogs bred in England.

The data must necessarily be of an approximate nature, because (i) if a bitch be lined only once there is less chance of obtaining a litter than if she be served twice, and in our experimental work the chief aim is to obtain a litter. The cost of keeping dogs which fail to litter, and there are many slips, is already too heavy for a poor institution. And (ii) the date of littering is that of the day when the bitch was found to have puppies. With our Pekinese and Pekinese hybrids we have noted a marked objection to littering in the presence of anyone; they cry and whine and will not attend to business†. The bitches as a rule litter during the night, most probably in the early morning, for this is the time the attendant usually finds that the bitch has just littered but has not yet cleaned up, or is just littering. Accordingly the date of littering is in this paper taken to be the day on which she is known to have littered, or the day on which she is found with puppies, although these might in some cases have been born actually twelve hours earlier.

As to the date of mating, when it has occurred twice, for the most part the mid-date between the two matings has been taken. By duration of pregnancy we understand accordingly here the time which has elapsed between this mid-date and the day on which the bitch is known to have littered, or has been found to have puppies. This is not of course the period of true pregnancy, for we do not know the time at which the spermatozoon comes into conjunction with the ovum, nor to a few hours the time of the littering, indeed the latter sometimes lasts several hours. But it is as close as we can get by aid of experimental work, not intended solely for the present investigation, and it is close enough to give results of value for practical breeding.

* Acknowledgment must be made of assistance from the Royal Society Government Grant received on several occasions during the course of these experiments.

† I have sat up in my early days hours with a bitch, but to no purpose. Half-an-hour after I had left her in despair, she would have her pups without more fuss! K. P.

The purpose of the experiments being to study hybridisation we had relatively few pure-bred dogs, and rarely bred Pekinese with Pekinese, or Pomeranian with Pomeranian. The hybrid was termed a "Pompek," and for shortness we may speak of Poms and Peks. It is possible that Peks in pure breeding and Poms in pure breeding have different durations of pregnancy, but our experiments are not adequate to determine a slight difference if it exists.

Pure Pom bitches had an average duration of 60·1 days of pregnancy, whatever the sire. When mated with Pom sires 60·2 days' duration. When mated with Pek sires 60·0 days. When mated with Pompeks 61·0 days. These are all on relatively few numbers, and the results do not suggest that the period of pregnancy of a pure Pom bitch is influenced by the race of the sire.

Turning to Peks the pure Pek bitch, whatever the sire, had an average duration of 61·4 days. When mated with a pure Pek the duration was 62·6 days, with a pure Pom 61·3, and with a Pompek 59·9 days. It may be asked why the sire should affect the period of pregnancy of the bitch? The answer is that the period of pregnancy is influenced by the nature of the litter, e.g. the larger the litter (and the heavier probably) the shorter is the pregnancy. It may be that the number of the litter depends entirely on the bitch, but it is not impossible that it depends in part also on the sire*. Hence it by no means follows that the duration of pregnancy will be the same with a cross and with a pure mating. Our results do not indicate any such relation in the averages for pure Pom bitches. More might be read into the case of the pure Pek bitches, but when we see that the duration of pregnancy can vary from 55 to 68 days, we are not inclined to lay any stress on differences such as the above, which have in fact probable errors of the order of one day. We shall see that the mean duration of pregnancy for all available material is 60·76 days, and we are not able on the basis of our material to lay any stress on the difference involved in a Pom 60·4 and a Pek 61·4 days' duration. It seems probable that the age of the bitch, the order of the pregnancy and the size of the litter have more to do with the duration of pregnancy than the race of bitch or dog in these small dogs.

(ii) One grave difficulty in our breeding work has been the lengthy period which in certain cases has elapsed before the bitch showed the smallest sign of heat. In one case 4½ years, and in seven cases three or more years out of a total of 54 first litters for which data were available. It will be observed how very much adds to the cost of experiments of this nature, if a mating which is desired has to be postponed for two or even more years. As a rule after the first pregnancy the bitch comes into season twice a year, but by no means at fixed intervals; to what extent these are varied by (i) the absence of mating†, (ii) the length of suckling, (iii) the failure to have a litter after mating, or (iv) the age of the bitch, has not been adequately determined. The period of suckling, 4 to 6 weeks, depends largely on the size of the litter and the age of the bitch, but also on the condition at littering

* There is some evidence to indicate that in man twinning may arise from the Father's side.

† In some cases it was thought desirable owing to the youth of the bitch, or her non-recovery after the previous litter to full health and strength, to allow the heat to pass without mating.

and the food she will consent to take*. As a rule, however, there is a season at the end of spring and another at the beginning of winter. The following is a typical example:

Setie: born Oct. 8, 1923. 1st heat, Oct. 1924; 2nd, June 1925; 3rd, Dec. 1925; 4th, July 1926; 5th, March 1927; 6th, Oct. 1927; 7th, June 1928; 8th, Nov. 1928; 9th, July 1929. She was mated on all nine occasions and gave birth to 29 puppies. She was parted with after the 9th litter.

Here is another illustration:

Meg bhan: born May 3, 1913. 1st heat, Feb. 1914; 2nd, Aug. 1914 (no litter); 3rd, March 1915; 4th, April 1916; 5th, Nov. 1916; 6th, June 1917; 7th, Nov. 1918; 8th, Feb. 1920; 9th, Oct. 1921. She had to be destroyed in 1922, having also given birth to 29 puppies. The occurrence of heat is here more irregular, but may reasonably be associated with difficulties as to food during the War.

One last case:

Siri: born July 28, 1915. 1st heat, Aug. 1916 (no litter); 2nd, March 1917; 3rd, Oct. 1917; 4th, May 1918; 5th, Nov. 1919; 6th, May 1920; 7th, June 1921. Total number of puppies born 23.

Irregularities chiefly occur when the bitch is very young or old, but a general discussion of the intervals between heats would require more data, especially with regard to suckling period and food, than our records provide.

(2) We will deal in the first place with the first litter, which usually, but not invariably, corresponds with the first heat and the first mating. We have only

TABLE I. *Age at First Littering and Duration of Pregnancy.*

Age of Bitch at First Littering (Central Values in Months).

Duration of Pregnancy in Days.	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	Totals
56	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
57	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	2
58	—	3	2	1	1	—	—	—	—	—	—	—	—	—	—	—	7
59	—	1	—	2	—	2	—	—	—	—	—	—	—	—	—	—	5
60	—	—	1	1	1	—	—	—	1	1	1	—	—	—	—	—	6
61	—	—	1	1	1	1	—	—	—	—	—	—	—	—	—	—	4
62	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	2
63	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	2
64	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
65	—	—	3	—	—	—	—	—	—	—	—	—	—	—	—	—	3
66	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
67	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	2
68	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	1
Totals	1	7	9	6	4	5	—	—	1	2	1	—	—	—	—	1	37

* Pekinese and Pekinese hybrids will often both before and after littering refuse cows' milk, or can only be induced to take it, if sponge cake be soaked with it!

TABLE II. *Age at First Littering and Size of Litter.*

Age of Bitch at First Littering (Central Values in Months)

Size of Litter.	*9*	*12*	*15*	*18*	*21*	*24*	*27*	*30*	*33*	*36*	*39*	*42*	*45*	*48*	*51*	*54*	Totals
1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
2	1	2	4	2	2	1	—	—	1	3	1	—	—	—	—	—	17
3	2	—	4	1	2	2	—	—	—	1	—	1	—	—	—	1	14
4	—	4	—	3	—	—	2	—	1	—	—	—	—	—	—	—	10
5	1	2	3	—	1	2	—	—	—	—	—	—	—	—	—	—	9
6	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	1
7	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	2
Totals	4	9	12	7	5	6	2	—	2	4	1	1	—	—	—	1	54

37 cases in which the duration of pregnancy is provided for the age of mother at first litter. We have 54 cases in which the number of puppies is known for age of mother at first litter (see Tables I and II).

The average age of the mother at first litter is 20·33 months* in the first table and 19·95 in the second table. The average duration of first pregnancy is 60·86 days, while the average duration of all pregnancies is 60·76 days. There is nothing very different in the first pregnancy as far as its average duration is concerned from the average duration of later pregnancies.

The number of puppies in the first litter averages 3·37, while the average number for all litters is 3·22. This does not, of course, prove that the first litter is the most numerous, but only that it has somewhat more than the average number of puppies. We shall return to this point later.

The constants of the two tables are as follows:

Table I. Mean age of Mother at first litter 20·33 months
Standard Deviation = 9·156 months
Mean Length of Pregnancy 60·86 days
Standard Deviation = 3·112 days
Correlation ·143 ± ·109

Table II. Mean age of Mother at first litter 19·95 months
Standard Deviation = 9·567 months
Mean number of Puppies 3·37
Standard Deviation = 1·365 puppies
Correlation −·347 ± ·081

The latter correlation is significant, the former cannot be said to be. The general meaning if both were significant would be that:

The older the bitch at first pregnancy the fewer puppies she will have, and the longer the pregnancy.

* By a "month" in this paper is to be understood an average calendar month of 30·4 days.

That the pregnancy is longer may merely arise from the fact that the litter is smaller, the fertility of the bitch depending upon her age. The following results are suggestive:

Age of Bitch in Months	Mean Length of 1st Pregnancy	Mean Number of Puppies
8—13	60·75	3·38
14—19	60·80	3·57
20—25	60·56	3·55
Above 25	62·40	2·81

The average age at first litter being almost exactly twenty months, and the duration of first pregnancy almost exactly two months, we conclude that in these bitches the first heat occurred on the average at 18 months with a variability of 9·4 months, the distribution of this onset of puberty being very skew.

(3) After the above consideration of the first pregnancy, based admittedly on very slender data, we turn to the general relations between the four variates: Age of Mother (a), Size of Litter (l), Order of Pregnancy (ω), and Duration of Pregnancy (d). Our data are arranged in the six correlation tables, Tables III—VIII, to be found on this and the following pages.

TABLE III. *Order of Pregnancy and Size of Litter.*

Size of Litter.	Order of Pregnancy. I	II	III	IV	V	VI	VII	VIII	IX	Totals
1	2	4	4	5	2	—	2	—	—	19
2	17	9	6	2	3	2	1	2	—	42
3	14	15	7	5	2	—	2	1	—	46
4	11	6	10	2	4	3	1	—	—	37
5	9	6	3	3	—	2	—	—	1	24
6	2	1	1	3	1	1	—	—	—	9
7	2	—	—	—	—	—	—	—	—	2
Totals	57	41	31	20	12	8	6	3	1	179

Table III provides the relation between the order of pregnancy and the size of the litter. Matings not followed by pregnancy are omitted; one first pregnancy which was a miscarriage, and one second pregnancy in which the total number of puppies born was not recorded, have been disregarded. Table III contains 179 pregnancies leading to 577 puppies. The following are the constants of the table:

$\bar{l}$ = Mean No. of Puppies = 3·22 ± ·071,

σ_l = Standard Deviation of No. of Puppies = 1·4084 ± ·050,

$\bar{\omega}$ = Mean No. of Pregnancies = 2·77 ± ·093,

σ_ω = Standard Deviation of No. of Pregnancies = 1·8402 ± ·066,

$r_{l\omega}$ = Correlation of Order of Pregnancy and Size of Litter = − ·0509 ± ·0503.

Accordingly if we could trust to the regression being linear, there would appear to be no significant relation between the order of the pregnancy and the size of the litter. We must accordingly investigate the means of each array. We have:

Order of Pregnancy	I	II	III	IV	V	VI—IX	Mean all Pregnancies
Size of Litter ...	3·39	3·10	3·16	3·25	3·00	3·22	3·22

Although this appears to make the litter at first pregnancy the largest, the actual value in this case is $3{\cdot}387 \pm {\cdot}126$, which does not indicate any significant difference from the general mean 3·223.

This result appears to contradict the ordinary impression, which we ourselves have shared, that the litters of the first and of the last one or two pregnancies are smaller than the average. The source of this apparent paradox may lie in the fact that we are not dealing with nine successive pregnancies of the same bitches; we are clubbing bitches of varied degrees of fertility together, and the horizontal margin shows that many bitches drop out after the first two or three matings. It was only those of the greater experimental interest that could be preserved to the last stages of their reproductive powers, and this was peculiarly the case during the War years, when, owing to the scarcity and cost of food, the sole aim was to keep enough dogs alive to continue the work when peace came*.

We pass next to the duration of the pregnancy and the size of the litter. We have already drawn attention to the fact that the dates of mating have not always been recorded. Further, there were not always two matings, and if there were, they might be on successive days, or there might be an intervening day. We have the following results according as we measure the pregnancy from the day of the first mating, from the day of the second mating or from the midday between:

	1st Mating	Midday	2nd Mating
Mean Duration of Pregnancy	61·41	60·76	59·74
Standard Deviation ...	3·0846	3·1825	3·1969

all in days.

It is clear that there was an interval of about 1·7 days between the two matings. The midday is not midway between the first and second matings, because the first mating includes all those cases in which there was only a single mating. Examining the standard deviations, it will be seen that the duration of pregnancy varies least about the mean duration from first mating to littering. It seems probable therefore that in most cases the first mating is successful. For practical purposes therefore we may say that a bitch of these breeds will litter after an interval of $61{\cdot}41 \pm 2{\cdot}14$ days from first mating, or that a bitch is very unlikely to have a litter at all if it

* Even at present the size of our Animal House and the extent of our funds do not permit of more than 15 to 20 adult dogs being kept at one time.

does not occur between the 55th and 68th* days from first mating, the 61st to 62nd days being the most probable days for littering. If there has been a double mating then the bitch will litter most probably 60·76 ± 2·15 days from the midday between the two matings. If a bitch does not litter between the 54th and 67th days from the midday of mating, she is very unlikely to have a litter†. Our actual experience has been one bitch littering on the 55th day and two on the 68th day.

TABLE IV. *Duration of Pregnancy and Size of Litter.*

Size of Litter.	Duration of Pregnancy in Days.														
	55	*56*	*57*	*58*	*59*	*60*	*61*	*62*	*63*	*64*	*65*	*66*	*67*	*68*	Totals
1	—	—	—	1	—	1	1	—	—	3	—	—	2	—	8
2	—	—	—	2	1	2	3	4	2	1	4	4	2	—	25
3	—	—	2	2	2	8	4	3	3	3	4	—	2	2	35
4	—	1	2	5	5	1	1	1	1	1	1	—	—	—	19
5	—	1	—	2	4	3	—	2	1	1	1	—	—	—	15
6	1	—	1	—	2	2	—	—	—	—	1	—	—	—	7
7	—	—	—	1	—	—	—	—	—	—	—	—	—	—	1
Totals	1	2	5	13	14	17	9	10	7	9	11	4	6	2	110

In Table IV the length of pregnancy is measured from the first mating and the constants of the table are as follows:

$\bar{l}$ = Mean Size of Litter = 3·300,
σ_l = Standard Deviation = 1·3655,
$\bar{d}$ = Mean Duration of Pregnancy = 61·409,
σ_d = Standard Deviation = 3·0846,
r_{ld} = Correlation of Size of Litter and Duration of Pregnancy = − ·4479 ± ·0514.

There is thus a significant and quite considerable negative correlation between size of litter and duration of pregnancy. This correlation may be illustrated by the following mean values:

Size of Litter	Mean Duration of Pregnancy in days
1	63·13
2	62·96
3	61·97
4	59·47
5	60·27
6	59·29
7	58·00

* This is based on plus and minus three times the probable error from the mean.

† In some cases a bitch after mating makes up her mind that she will litter, she develops, and sometimes shows signs of milk, and finally may even prepare her lair, without having any puppies.

The corresponding prediction formulae are:

d_l = probable duration of pregnancy for given size of litter $l = 64{\cdot}75 - 1{\cdot}012\,l$,

or pregnancy is delayed about one day for each decrease of one in the litter.

l_d = probable litter for given duration of pregnancy $d = 15{\cdot}48 - {\cdot}198\,d$,

or five days' delay in littering would on the average denote a reduction of two in the litter.

We now turn to Table V, which gives the relation between the duration and order of pregnancy.

TABLE V. *Duration of Pregnancy and Order of Pregnancy.*

Duration of Pregnancy in Days.

Order of Pregnancy.	*55*	*56*	*57*	*58*	*59*	*60*	*61*	*62*	*63*	*64*	*65*	*66*	*67*	*68*	Totals
I	—	1	2	7	5	6	4	2	2	1	3	1	2	1	37
II	—	1	2	3	2	3	—	6	2	3	4	2	1	—	29
III	—	—	—	2	2	4	3	1	—	2	1	1	1	—	17
IV	—	—	—	1	3	1	1	1	—	1	1	—	1	—	10
V	1	—	1	—	1	2	—	—	—	1	—	—	—	—	6
VI	—	—	—	—	1	1	1	—	1	—	—	—	—	—	4
VII	—	—	—	—	—	—	—	—	1	1	2	—	1	—	5
VIII	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
IX	—	—	—	—	—	—	—	—	1	—	—	—	—	—	1
Totals	1	2	5	13	14	17	9	10	7	9	11	4	6	2	110

The constants of the table are as follows:

$\bar{d}$ = Mean Duration of Pregnancy = $61{\cdot}409 \pm {\cdot}198$,

σ_d = Standard Deviation of $d = 3{\cdot}0849 \pm {\cdot}1403$,

$\bar{\omega}$ = Mean Order of Pregnancy = $2{\cdot}6546 \pm {\cdot}1187$,

σ_ω = Standard Deviation of $\omega = 1{\cdot}8461 \pm {\cdot}0839$,

$r_{d\omega}$ = Correlation of Duration with Order of Pregnancy = $-{\cdot}1780 \pm {\cdot}0623$.

There is thus a positive correlation between the order of pregnancy and its duration; it is rather small but is probably significant. As the regression is unlikely to be linear we determined the correlation ratio of duration of pregnancy on order of pregnancy and found

$$\eta_{d.\omega} = {\cdot}3785,$$

indicating an association more than double that determined for the correlation coefficient.

Our data are too scant to give a close approximation to the manner in which the duration changes with the order of pregnancy, but the following series of mean durations:

Order of Pregnancy	Mean Duration
I	60·86
II	61·76
III	61·47
IV	61·40
V	59·17
VI	60·75
VII—IX	64·71
All Pregnancies	61·41

suggest that the first pregnancy has a duration rather below the mean; the duration rises above the mean in the second and third pregnancies, sinks below the mean again in the fourth and fifth to become very protracted in the extreme pregnancies. This is only a *suggestion*, but it seems not out of accord with probable physiological changes.

The partial correlations $r_{\omega d.l}$, $r_{\omega l.d}$ and $r_{ld.\omega}$ are not without some interest, although they will not bear much stressing. Thus

$$r_{\omega d} = \cdot 1780, \text{ but } r_{\omega d.l} = \cdot 1738,$$

and we see that the observed relation between the order and duration of pregnancy is little influenced by the fact that the duration depends upon the size of the litter. Again,

$$r_{\omega l} = -\cdot 0509, \text{ but } r_{\omega l.d} = +\cdot 0323;$$

accordingly such little relation as there exists between the order of the pregnancy and the size of the litter is reversed, or is practically zero, when we take a constant duration of pregnancy.

Finally,

$$r_{ld} = -\cdot 4479, \text{ but } r_{ld.\omega} = -\cdot 4465;$$

thus the association of a long duration of pregnancy with a small litter is practically independent of the order of pregnancy. These are all points concerning which it would be desirable to collect more ample data.

(4) We will now consider what effect the age of the bitch has on the size of the litter and the duration of pregnancy; it will clearly be of necessity fairly highly correlated with the order of pregnancy. Now the age of the bitch may be considered with relation to the mating or the littering. Table VI[A] (see p. 317) provides the relation of the size of the litter (l) to the age of the bitch (a) at mating, and Table VI[B] that of the size of the litter with the age of the mother at littering (a').

The constants of these two tables are given below:

Table VI[A].

$\bar{a} = 34{\cdot}205 \pm 1{\cdot}241$ months,
$\sigma_a = 19{\cdot}4772 \pm {\cdot}8778$ months,
$\bar{l} = 3{\cdot}277 \pm {\cdot}088$,
$\sigma_l = 1{\cdot}3772 \pm {\cdot}0620$,
$r_{al} = -{\cdot}1722 \pm {\cdot}0618$.

Table VI[B].

$\bar{a}' = 37{\cdot}115 \pm 1{\cdot}040$ months,
$\sigma_{a'} = 19{\cdot}2670 \pm {\cdot}7462$ months,
$\bar{l} = 3{\cdot}160 \pm {\cdot}074$,
$\sigma_l = 1{\cdot}3753 \pm {\cdot}0525$,
$r_{a'l} = -{\cdot}1468 \pm {\cdot}0528$.

TABLE VI[A]. *Age of Bitch at Date of Mating and Size of Litter.*

Age of Bitch at Date of Mating (Central Values in Months).

Size of Litter.	*6*	*9*	*12*	*15*	*18*	*21*	*24*	*27*	*30*	*33*	*36*	*39*	*42*	*45*	*48*	*51*	*54*	*57*	*60*	*63*	*66*	*69*	*72*	*75*	*78*	*81*	Totals
1	—	—	—	—	1	—	—	1	—	—	—	—	—	—	1	—	—	—	1	2	—	1	1	—	1	—	9
2	2	2	2	2	2	1	—	—	2	3	—	2	1	2	—	1	—	—	—	1	—	1	—	1	—	1	26
3	—	—	2	4	3	5	1	3	—	1	4	2	2	—	—	3	—	1	1	1	—	1	—	—	—	—	34
4	—	3	—	3	—	—	—	3	1	1	1	—	2	—	1	1	1	1	—	—	2	—	—	—	—	—	20
5	—	2	2	—	1	4	1	—	1	1	—	—	—	—	—	1	—	—	—	—	1	1	—	—	—	—	15
6	—	—	—	—	—	1	—	1	—	1	—	—	—	1	1	—	—	2	—	—	—	—	—	—	—	—	7
7	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	2	7	7	9	7	11	2	8	4	7	5	4	5	3	3	6	1	4	2	4	3	4	1	1	1	1	112

TABLE VI[B]. *Age of Bitch at Date of Littering and Size of Litter.*

Age of Bitch at Date of Littering (Central Values in Months).

Size of Litter.	*9*	*12*	*15*	*18*	*21*	*24*	*27*	*30*	*33*	*36*	*39*	*42*	*45*	*48*	*51*	*54*	*57*	*60*	*63*	*66*	*69*	*72*	*75*	*78*	*81*	*84*	*87*	*90*	*93*	Totals
1	—	1	—	1	1	—	—	1	2	—	—	1	1	—	1	1	—	1	2	1	—	1	1	1	—	—	—	—	—	17
2	2	2	3	1	3	1	—	1	5	4	3	2	2	2	—	1	1	—	—	1	—	1	—	1	—	1	—	—	1	38
3	1	—	3	5	3	6	5	—	—	4	2	1	2	—	2	1	—	1	1	1	—	1	1	—	1	—	—	—	—	41
4	1	3	1	3	—	—	2	2	1	1	2	4	2	1	3	1	2	1	—	—	2	1	—	—	—	—	—	—	—	33
5	1	3	1	—	2	4	2	—	1	1	—	—	—	—	—	1	—	—	—	—	2	—	—	—	—	—	—	—	—	18
6	—	—	—	—	—	1	—	1	1	1	—	—	1	—	1	—	—	2	—	—	—	—	—	—	—	—	—	—	—	8
7	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
Totals	5	9	9	10	9	12	9	5	10	11	7	8	8	3	7	5	3	5	3	3	4	4	2	2	1	1	—	—	1	156

The relation between the age at mating and the size of the litter, the number of puppies being smaller the older the bitch, is probably significant, but is not very considerable; it is larger than the correlation between order of pregnancy and size of the litter (−·0509). It is probably reduced in experimental breeding because when the bitch's fertility is reduced, i.e. when she, although mated, produces no litter or only one or two puppies, she is discarded for stud purposes*. The difference between the bitches' ages at mating and pregnancy = $\bar{a}' - \bar{a} = 2{\cdot}910$ months = 88·5 days. This is *not* the average duration of pregnancy because the second series of dogs is not identical with the first, there are 44 additional entries principally due to the records of C. H. Usher, which provide dates of littering but not those of mating. Even allowing for an average period of 60·8 days for pregnancy, it will be seen that the Aberdeen dogs were on the whole mated to greater ages than the London dogs. As to the remainder of the constants there is no difference of practical importance between them. Accordingly, as the only advantage of taking age of bitch at litter over age at mating lies in the increase of entries, and as this involves a risk of heterogeneity (as Usher introduced new Pom blood while Pearson, after the first cross of Pekinese with Pompeks, continued to inbreed), we shall for the remainder of this paper confine our attention to Age of Mother at Mating.

Table VII (p. 320) shows the relationship between Age of Bitch at mating and Order of Pregnancy. The very appearance of the table indicates how considerable the correlation is, a result which it was easy to predict.

The constants of Table VII are as follows:

$\bar{a}$ = Mean Age at Mating = 34·289 ± 1·237 months,
σ_a = Standard Deviation of Age = 19·5756 ± ·8744 months,
$\bar{\omega}$ = Mean Order of Pregnancy = 2·632 ± ·1167,
σ_ω = Standard Deviation of Order = 1·8461 ± ·0825,
$r_{a\omega}$ = Correlation of Age and Order = ·7967 ± ·0231.

The following table shows the average age at each pregnancy:

Order of Pregnancy	Observed Age	Smoothed Values from Regression Line
I	17·76 months	20·50 months
II	29·36 ,,	28·95 ,,
III	40·65 ,,	37·40 ,,
IV	51·33 ,,	45·85 ,,
V	53·50 ,,	54·29 ,,
VI	57·60 ,,	62·74 ,,
VII	64·80 ,,	71·19 ,,
VIII and IX	68·00 ,,	83·86 ,,

That the observed ages at later pregnancies fall so much below those calculated from the correlation formula is no doubt due to the fact that the more fecund bitches had litters earlier and rarely missed a mating. There are, however, only

* Matings leading to no litters have been excluded from these tables. It is not possible in such cases to determine whether the dog or bitch is at fault.

TABLE VII. *Age of Bitch at Date of Mating and Order of Pregnancy.*

Age of Bitch at Date of Mating (Central Values in Months).

Order of Pregnancy.	*6*	*9*	*12*	*15*	*18*	*21*	*24*	*27*	*30*	*33*	*36*	*39*	*42*	*45*	*48*	*51*	*54*	*57*	*60*	*63*	*66*	*69*	*72*	*75*	*78*	*81*	Totals
I	2	7	7	8	2	5	—	—	2	1	2	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	38
II	—	—	—	2	5	6	1	3	1	1	2	1	3	1	—	—	—	1	—	—	—	1	—	—	—	—	28
III	—	—	—	—	—	—	1	5	1	3	1	—	1	1	1	2	—	—	1	2	1	—	—	—	—	—	20
IV	—	—	—	—	—	—	—	—	—	2	—	2	—	1	—	—	—	—	—	2	—	1	—	—	1	—	9
V	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	2	—	1	—	—	—	—	1	—	—	—	6
VI	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	1	—	—	2	—	—	—	—	—	5
VII	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	1	—	1	—	—	—	1	5
VIII	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1	—	—	2
IX	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1
Totals	2	7	7	10	7	11	2	8	4	7	5	4	5	3	3	6	1	4	2	5	3	4	1	1	1	1	114

TABLE VIII. *Age of Bitch at Date of Mating and Duration of Pregnancy.*

Age of Bitch at Date of Mating (Central Values in Months).

Duration of Pregnancy in Days.	*6*	*9*	*12*	*15*	*18*	*21*	*24*	*27*	*30*	*33*	*36*	*39*	*42*	*45*	*48*	*51*	*54*	*57*	*60*	*63*	*66*	*69*	*72*	*75*	*78*	*81*	Totals
54	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	1
55	—	—	—	—	—	—	—	—	—	—	0·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0·5
56	—	—	2	—	0·5	—	—	—	—	—	0·5	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	4
57	—	3·5	—	0·5	0·5	2·5	—	0·5	1	0·5	—	—	2	—	—	1·5	—	1	—	—	—	—	—	—	—	—	13·5
58	—	0·5	1	1·5	1	1·5	—	1	0·5	1	—	1	—	0·5	—	0·5	—	—	—	—	0·5	—	—	—	—	—	10·5
59	—	1	2	2	—	2	—	1·5	1	0·5	1	—	1	0·5	—	1	—	1	—	—	1·5	—	—	—	—	—	16
60	—	—	—	2	1	0·5	—	2	0·5	1	—	1	—	—	—	—	—	—	—	—	—	—	1	—	—	—	9
61	—	—	—	—	1	2	—	1	—	1	1	—	—	1	—	1	—	—	—	1	1	2	—	—	—	—	12
62	—	1	—	3	—	1·5	—	—	1	—	—	—	—	—	0·5	—	—	—	—	—	—	1	—	—	0·5	0·5	9
63	—	—	—	—	2	—	1	0·5	—	—	—	—	—	—	0·5	0·5	1	—	1	—	—	—	—	—	0·5	0·5	7·5
64	1	—	2	—	1	—	—	1·5	—	1	—	—	2	1	—	0·5	—	—	—	0·5	—	—	—	0·5	—	—	11
65	—	0·5	—	—	—	—	—	—	—	0·5	1	0·5	—	—	—	—	—	1	—	1·5	—	—	—	0·5	—	—	5·5
66	—	0·5	—	—	—	—	1	—	—	0·5	1	0·5	—	—	—	—	—	—	—	1	—	—	—	—	—	—	4·5
67	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	0·5	—	—	—	—	—	—	—	2·5
68	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0·5	—	—	—	—	—	—	—	0·5
Totals	2	7	7	9	7	11	2	8	4	6	5	3	5	3	2	5	1	4	2	4	3	3	1	1	1	1	107

three bitches who reached the VIII and IX pregnancies, and these count very little in the determination of the correlation. The correlation-ratio of ω on a does not differ sensibly from $r_{\omega a}$.

The average interval between pregnancies is 8·45 months. The physiological interval is about 6 months, and the observed increase is due to matings which were omitted or failed when the bitch was in season. It has been observed also that an aged bitch may occasionally omit one or more heats.

The regression equation giving the probable age ($\check{a}$) at a given pregnancy ω is

$$\check{a} = 12{\cdot}054 + {\cdot}8{\cdot}448\omega.$$

The other constants of Table VII, considering how the total numbers vary from table to table owing to one or another omission in the record, are in reasonable accordance with those of Tables III and V.

We now turn to Table VIII, associating the Age of the Bitch at mating with the Duration of Pregnancy.

The constants of this table are as follows:

$\bar{a}$ = Mean Age of Bitch at Mating = 33·56 ± 1·272 months,
σ_a = Standard Deviation of Age = 19·5119 ± ·8997 months,
$\bar{d}$ = Duration of Pregnancy = 60·66 ± ·198 days,
σ_d = Standard Deviation of Duration = 3·0434 ± ·1403 days,
r_{ad} = Correlation between Age at Mating and Duration of Pregnancy = ·1547 ± ·0636.

The first four constants are within their probable errors of the like characters previously determined. The correlation is small but probably just significant. It is noteworthy that while the correlations of duration of pregnancy with age and with order of pregnancy are both small and positive the latter appears to be somewhat the larger.

The difficulty, however, with practical breeding lies in the economic factor, that matings in the case of such expensive animals as dogs will no longer be made (unless the bitch is of especial value or interest) after the fecundity has begun seriously to diminish.

The regression equation of Duration of Pregnancy on Age at Mating is

$$\check{d} = {\cdot}02413a + 59{\cdot}85.$$

Hence if we take the lowest age of the first heat at 9 months and the highest age of last pregnancy at 84 months = 7 years*, we have for the corresponding durations 60·07 and 61·88 days, or age would have a maximum range of influence of 2 days only on period of pregnancy.

If we take the correlation of Age and Duration of Pregnancy for constant Order, and the correlation of Order and Duration of Pregnancy for constant Age, we have

$$r_{ad.\omega} = {\cdot}0111,$$
$$r_{\omega d.a} = {\cdot}0918,$$

* Our experience seems to show that these dogs have on *the average* a life of nine years or even less, and that few bitches are of use for breeding purposes beyond six or seven years.

and these seem to indicate, taken at their face values, that the number of the pregnancy is more important than the age of the bitch for the duration. But the data are too slender, and the artificial selection of bitches for stud purposes too great, for any stress—other than that of suggestion—to be placed on this result.

(5) *Conclusions.* It seems worth while publishing these results. It is true that the disappearance during the War of the mating books of the Scottish bred dogs, before the dates of mating had been recorded on the schedules, has much reduced the available material. Further, the experiments were not made directly to determine problems regarding gestation in dogs; their primary purpose was to investigate as economically as possible the inheritance of certain characters in dogs.

Thus the bitches were not retained in the kennels long after their period of maximum fecundity was passed. Again, in London kennels with only yard exercise, general fitness is far less than can be maintained in the country, or even in a London home with daily walks. However, the general results seem suggestive enough to make further research worth while. The principal correlations are:

$$r_{dw} = +\cdot1780,$$
$$r_{ld} = -\cdot4479, \quad r_{lw} = -\cdot0509, \quad r_{la} = -\cdot1722,$$
$$r_{aw} = +\cdot7967, \quad r_{ad} = +\cdot1547.$$

Thus the three factors, increasing duration of pregnancy, increasing number of pregnancies, increasing age, all tend to decrease the size of the litter. In the first case it is probable that it is the size of the litter which is the causal factor and hastens the end of gestation. This gives the most marked correlation, and it would be of interest to determine—size of litter being associated with weight—whether in other mammals the average period of gestation is less for male than for female offspring, and less for twins than for single births. We have seen that the correlation coefficient between the duration and order of pregnancy is small, because the relationship is not linear. It is hardly possible to account for the small coefficient of age and size of litter on similar grounds*, but it may be possible to do so on the ground of artificial selection. Probably the fertility of the bitch is not diminished until she is over five years of age. Further, we cannot attribute the small relationship between age and duration of pregnancy to markedly curved regression†. The

* The relationship is as follows:

Age in months	6—22	23—40	41—58	59—82
Size of Litter	3·42	3·30	3·64	2·41

† There is a fairly continuous increase thus:

Age in months	6—16	17—28	29—46	47—64	65 and over
Duration of Pregnancy	60·06	60·63	60·69	61·03	61·20

multiple-regression equation of Duration of Pregnancy on Order of Pregnancy and Age at start of Pregnancy is

$$\frac{d-\bar{d}}{\sigma_d} = \cdot 1499\,\frac{\omega-\bar{\omega}}{\sigma_\omega} + \cdot 0353\,\frac{a-\bar{a}}{\sigma_a},$$

which indicates that with equally likely deviations from the mean order of pregnancy and mean age, the former, the order of the pregnancy, will be more than four times as influential as the age *.

The fact remains that none of the factors we have taken into consideration suffices to provide a causal explanation for the duration of pregnancy varying from 55 to 68 days. Can it be that this duration is individual and possibly an inherited character? If so it would be of evolutionary importance. The evidence as to this possibility must be discussed on another occasion.

No one can recognise more clearly than the writers the paucity of their data, but this field of investigation is of considerable interest. It is possible that an appeal to large breeders of dogs might produce more ample data as the variates we are dealing with must have been recorded in many cases. We shall be content if the present paper leads others to collect and reduce material on a wider basis, dealing if possible with small dogs of a single species; for comparative purposes Pekinese or Pomeranians would be most serviceable.

* The actual numerical equation to determine the probable duration of pregnancy $\check{d}$, in days, for a bitch in her ωth pregnancy and of age a months is

$$\check{d} = 60{\cdot}557 + {\cdot}2505\omega + {\cdot}005{,}581a.$$

Thus a bitch in her fifth pregnancy and four years old—i.e. $\omega = 5$ and $a = 48$—would have a probable duration of pregnancy $\check{d}$ given by

$$\check{d} = 60{\cdot}557 + 1{\cdot}2525 + {\cdot}2679 = 62{\cdot}08 \text{ days}.$$

ON THE ASYMMETRY OF THE HUMAN SKULL.

By T. L. WOO, Ph.D. Lond., Research Fellow of the China Foundation for the Promotion of Education and Culture.

(1) Much has been written about the quantitative asymmetry of the brain, on the assumption that differentiated functioning of the right and left hemispheres might (or must) be manifested by differentiated *size*. On such a hypothesis the bony skull developing so as to fit the growing brain should exhibit significant evidence of this asymmetry. Reasoning in this way there is nothing in the least absurd in the fundamental conceptions of phrenology. What has been the misfortune of that science was the premature localisation of certain mental and sensory activities before any adequate statistical evidence was forthcoming (or had at least been published) for each such local assignment. Since in the case of a sensory or mental activity we might on the above hypothesis anticipate an exaggerated or at least a marked development of the brain and a correlated development of the skull, the question of the asymmetry of the latter becomes one of great importance. If it be possible to demonstrate that some well-used mental or sensory activity is controlled generally from a centre on one side of the brain and there is no correlated increase in skull size in that region, i.e. that there is no resulting asymmetry, we shall have a strong argument—it may not necessarily be a conclusive one—that this emphasised local brain activity is not highly correlated with *size*. To the same extent we weaken the standpoint of the phrenologist that a cranial "bump" which to a large extent connotes asymmetry* marks the special development of a local centre of brain activity.

(2) Most measurements of the skull have hitherto been taken in the service of anthropology, i.e. with a view to finding the differentiated characteristics of various races. Racial differences were first approached from the standpoint of appearance, in other words from the conception of portraiture. Anthropometricians endeavoured to give quantitative value to the differences that were obvious to them at first sight. They observed the roundness of the head, the breadth of the forehead, the height of the face, the ellipticity of the orbit and so forth. Such measurements are usually composite, covering more than one bone of the skull, and are generally far from suitable for testing the asymmetry of the skull. Indeed they often cover *both* the homologous bones the difference in the sizes of which leads to the asymmetry, or again are worthless for our purpose because the measurements are taken in the median sagittal plane.

* Always supposing that the homologous region—i.e. from the phrenologist's standpoint an independent mental or emotional trait—does not chance to be equally developed.

However valuable the present measures of the anthropologists may be for the purpose for which they were devised, it is clear that they can give no final answer to many important problems, and one of these is the asymmetry of the skull. For this purpose we need measurements on the individual bones of the skull, taken in homologous pairs. A special advantage of taking measurements on the individual bones of the skull is that we thus get some idea of the size and shape of relatively small regions, not indeed coinciding with the phrenological areas, but giving us a better appreciation of *local* asymmetries than the run of anthropometric measurements can. I think it might be useful to distinguish the two types of measurements as ethnometric and morphometric, for both are actually anthropometric. The division really refers to the purposes which they are to serve; for while some few ethnometric characters have morphometric value, the bulk of the latter could be used for ethnometric distinctions, and will undoubtedly be more and more so used in the future developments of anthropometry, i.e. we shall gradually come to the study of the ethnic differences of the individual bones of the skull, rather than those of its composite characters—just as a study of the individual long bones has greater ethnic value than a study merely of stature.

(3) Having need for one of the important morphometric problems to which I have referred to study the characters of the individual cranial bones, I took a number of measurements of each of these. I did this on the long series of Egyptian skulls, 26th to 30th dynasties (Series E), in the Biometric Laboratory, confining my attention to those classed as male, amounting to about 800 in number*. On the separate bones I took 63 measurements, partly chordal and partly arcual. Of these 63 measurements, 50 were corresponding measurements on homologous bones, and accordingly of value for determining the degree of asymmetry in the two sides of the skull, and for measuring what regions were in excess on the right or on the left side with the amount of that excess.

The following are the 25 measurements which were taken bilaterally (Figs. 1 and 2):

(*a*) *Frontal Bone.* F_1 = minimum arc from a point on the coronal suture equidistant from the bregma and stephanion to the upper border of the orbit immediately outside the supra-orbital notch. The line of the coronal suture is marked in pencil to indicate its general direction, so no account is taken of a local indentation in determining the terminal of this measurement. The point equidistant from the bregma and stephanion can be found with the aid of coordinate callipers, or with small dividers. Although the supra-orbital notch is very variable in form, there is no difficulty in making the steel-tape pass immediately outside it.

F_2 = arc from ophryon to stephanion. The ophryon is defined, for this purpose, to be the intersection of the minimum arc from nasion to bregma and the minimum arc (marked in pencil) between the temporal lines.

* Some 53,400 measurements were taken, and as each skull took over an hour to measure, it required a year's work to complete the series.

(*b*) *Parietal Bone.* P_2 = arc from bregma to sphenion* along the line of the coronal suture.

P_3 = minimum arc from bregma to asterion.

P_4 = minimum arc from sphenion to lambda, avoiding the temporal squama so that the arc falls entirely on the parietal bone†. This measurement is generally close to, but not identical with, the geodesic line.

(*c*) *Occipital Bone.* O_7 = arc from lambda to asterion along the line of the lambdoid suture. This may diverge appreciably from the geodesic line between the points.

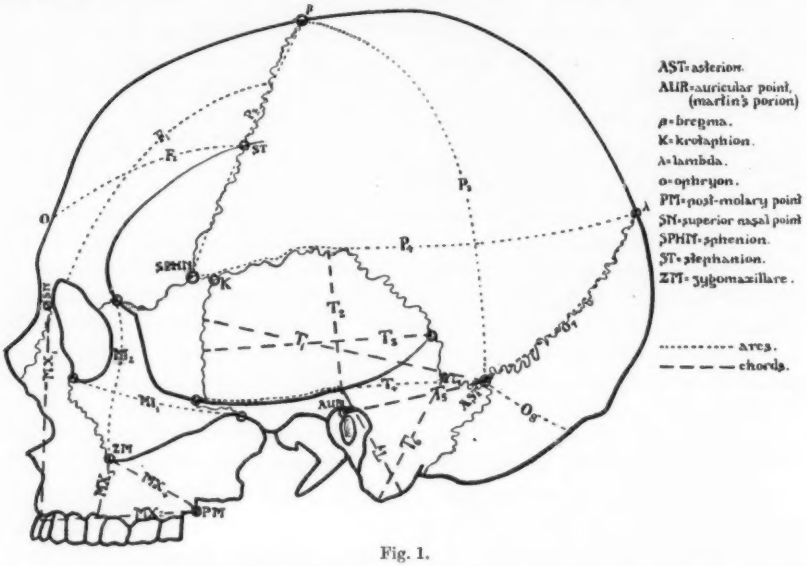

Fig. 1.

O_8 = arc from the median line of the occipital bone to the asterion. The median line is defined, for this purpose, by the minimum arc from opisthion to lambda, and the join of this line with the geodesic between the asteria gives the terminal. The measurement is then taken along that geodesic. The opisthion is here defined to be the point where the extension of the external occipital crest meets the border of the *foramen magnum.*

* If there be an epipteric bone at the pterion in contact with the frontal bone, the sphenion is supposed indeterminate.

† If there be an ossicle of the bregma, that "point" is accepted as the intersection of the lines (traced in pencil) marking the general direction of the coronal and sagittal sutures. The lambda and asterion are defined in a similar way, if supernumerary bones are present, though less exactly since each is defined to be the join of three sutural lines and, if it is necessary to continue them, they may not meet in a unique point. If the sutures round the lambda are very complex it may be necessary to use the same method.

O_9 = chord from basion to asterion.

(*d*) *Temporal Bone.* T_1 = maximum chord from the asterion to the anterior border of the temporal bone. When the anterior border—i.e. the spheno-squamous suture—is deeply dentated, but not otherwise, the anterior terminal is taken on the pencil line which marks its general direction. The point appears to be invariably above the zygomatic arch and it may be close to the pterion.

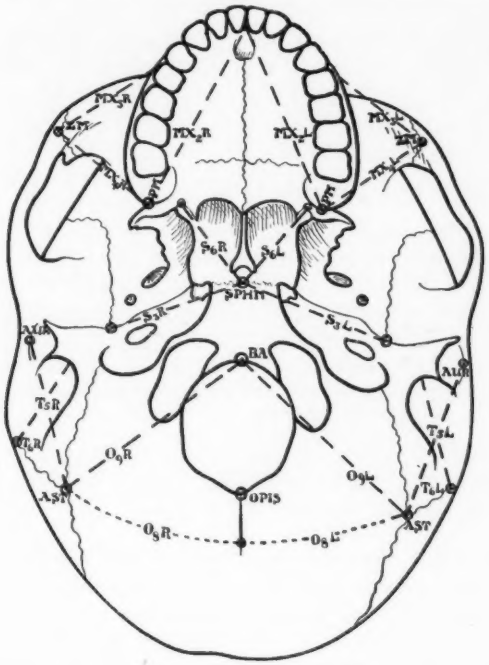

Fig. 2.

T_2 = chord from the auricular point to the point where the minimum arc from the auricular point to the bregma meets the upper border of the temporal squama. In doubtful cases only (as when the margin is slightly broken, or when there is a clear spinous process) the squamous border is marked in pencil to indicate its general direction. The auricular point is defined to be the point on the upper margin of the auricular passage which lies in the plane bisecting the orifice transversely*. It can generally be found in practice by continuing forward the curve of the thin lip of bone which terminates posteriorly in a well-marked notch on the upper part of the posterior wall of the passage.

* This point is Martin's "porion."

T_3 = maximum chord from the point where the backward extension of the temporal ridge meets the parietal bone to the anterior border of the squama. This measurement is not entirely satisfactory, but it would be difficult to devise a better measure of the antero-posterior length of the temporal squama. The temporal ridge is generally blunt, and in continuing it as a pencil line to meet the parieto-squamous suture considerable differences might be made by different workers*. When the spheno-squamous suture is deeply dentated, but not otherwise, the anterior terminal is taken on the pencil line which marks its general direction.

T_4 = minimum arc from asterion, above the auricular passage, along the upper border of the zygomatic ridge to the suture with the malar bone. This passes through the point on the temporal ridge, at the root of the zygomatic process, which is in the plane bisecting the auricular orifice transversely, i.e. the "auriculare" of Martin, the "point sus-auriculaire" of the French.

T_5 = chord from the asterion to the auricular point.

T_6 = maximum chord from a point on the suture with the parietal bone, equidistant from the asterion and the point where the temporal ridge meets the parietal bone, to the tip of the mastoid process. The suture in question is made up of parieto-squamous and parieto-mastoid portions and it is often irregular. The terminal is a point on the pencil line which indicates its general direction without regard to local indentations. Its position is somewhat uncertain owing to the fact that the point where the temporal ridge meets the parietal bone in some cases cannot be found precisely (see the definition of T_3).

T_7 = maximum chord from Martin's "auriculare" (see the definition of T_4) to the most remote part of the mastoid process. The mastoid terminals of the measurements T_6 and T_7 are not coincident.

(*e*) *Maxilla.* Mx_1 = chord from the point where the frontal, nasal and maxillary bones meet (the superior nasal point) to the lowest point on the alveolar process between the central incisors. If the alveolar processes of the two maxillae are completely fused the lower terminal will coincide with the alveolar point, but if they are slightly separated at the tips, as is often found, the two points will be distinct.

Mx_2 = chord from the lowest point on the alveolar process between the central incisors (as for Mx_1) to the "postreme point" on the alveolar process behind the last molar—the "post-molary point." The last molar is normally the third, but fully adult specimens for which no third molars have erupted are measured. The measurement cannot be taken, however, if the third molar was lost before death, or if the alveolar process was appreciably deformed by the loss of other teeth.

Mx_3 = chord from the lowest point on the malar-maxillary suture (Martin's "zygomaxillare") to the mid-point of the alveolar margin of the second premolar.

* For the purpose of the present study male skulls only were dealt with and the point in question would certainly be more difficult to determine on female specimens.

Mx_4 = chord from the lowest point on the malar-maxillary suture to the post-molary point.

(*f*) *Malar Bone.* Ml_1 = minimum arc from the point where the malar-maxillary suture crosses the lower border of the orbit to the lowest point on the zygomatic suture which is still on the lateral surface of the arch.

Ml_2 = minimum arc from the point where the malar ridge meets the fronto-malar suture to the lowest point on the malar-maxillary suture.

(*g*) *Sphenoid Bone.* S_2 = chord from the point where the frontal, sphenoid and temporal bones meet (Martin's "krotaphion") to the point in the median sagittal plane on the union of the basi-occipital and sphenoid bones (Martin's "spheno-basion.") The synchondrised basal suture can be marked by a pencil line with a close approach to accuracy in most cases.

S_3 = chord from the most posterior point of the sphenoid exposed on the base of the skull to the spheno-basion. The point is on the *spina angularis* which occupies the angle between the petrous and squamous portions of the temporal bone. This process is extremely variable in form, but the most posterior point on it can almost invariably be found without ambiguity.

S_5 = chord from the postreme point of the sphenoid exposed on the base of the skull to the krotaphion.

S_6 = chord from the spheno-basion to the lowest point on the suture between the medial pterygoid plate and the palate bone.

By a *minimum* arc a geodesic is to be understood. By a "suture" when much indented is to be understood a smooth line drawn midwise across the indentations. Arcs were measured to the nearest half millimetre and chords to the nearest $\frac{1}{10}$th millimetre.

(4) Having reduced my measurements I computed the means, standard deviations and coefficients of variation of each of the 50 measurements with their probable errors; also the coefficients of correlation of each of the 25 pairs of homologous measurements. The latter I obtained in two different ways as a check on my results, namely (i) by the usual product moment method for which the 25 correlation tables are given below, and (ii) by the well-known formula:

$$r_{xy} = \frac{\sigma_x^2 + \sigma_y^2 - \sigma^2_{x-y}}{2\sigma_x\sigma_y},$$

which involves a knowledge of the standard deviation of the difference of the two characters. The two methods should give identical results if we do not group x, y and $x - y$, but doing so and correcting for grouping we get slight differences in our results for r_{xy}. The two methods, however, give results sufficiently close to check the arithmetic.

Table I (p. 330) contains the values of the constants thus determined, the units being millimetres. In Table II (p. 333), I have arranged in order of significance those measurements in which the right and the left sides respectively are dominant.

TABLE I. *Constants of the Distributions and*

Measurements in millimetres.

Bone	Measurement	No.	Means	Standard Deviations	Coefficients of Variation	Correlation Coefficients *
			Constants of the Distributions			
Frontal	F_1 R	885	98·4381 ± ·1156	5·0988 ± ·0817	5·1797 ± ·0833	·9059 ± ·0041
	F_1 L		98·0426 ± ·1146	5·0567 ± ·0811	5·1577 ± ·0829	·9043 ± ·0041
	F_2 R	887	88·0070 ± ·1083	4·7796 ± ·0765	5·4309 ± ·0872	·9384 ± ·0027
	F_2 L		87·4366 ± ·1127	4·9760 ± ·0797	5·6910 ± ·0914	·9366 ± ·0028
Parietal	P_2 R	754	112·5789 ± ·1457	5·9322 ± ·1030	5·2694 ± ·0918	·6738 ± ·0134
	P_2 L		111·2487 ± ·1400	5·7017 ± ·0990	5·1251 ± ·0893	·6726 ± ·0134
	P_3 R	873	165·3960 ± ·1351	5·9180 ± ·0955	3·5781 ± ·0578	·7112 ± ·0113
	P_3 L		163·1612 ± ·1380	6·0431 ± ·0975	3·7038 ± ·0599	·7100 ± ·0113
	P_4 R	738	177·0305 ± ·1451	5·8437 ± ·1026	3·3010 ± ·0580	·7909 ± ·0093
	P_4 L		175·6199 ± ·1480	5·9608 ± ·1047	3·3942 ± ·0597	·7897 ± ·0093
Temporal	T_1 R	866	86·9157 ± ·0892	3·8921 ± ·0631	4·4780 ± ·0727	·8133 ± ·0078
	T_1 L		86·0242 ± ·0879	3·8341 ± ·0622	4·4570 ± ·0724	·8105 ± ·0079
	T_2 R	856	46·6262 ± ·0872	3·7813 ± ·0616	8·1098 ± ·1331	·8145 ± ·0078
	T_2 L		46·7956 ± ·0855	3·7090 ± ·0605	7·9260 ± ·1300	·8116 ± ·0079
	T_3 R	871	66·2204 ± ·0976	4·2700 ± ·0690	6·4482 ± ·1046	·8503 ± ·0064
	T_3 L		65·7371 ± ·0972	4·2537 ± ·0687	6·4709 ± ·1050	·8480 ± ·0064
	T_4 R	726	99·7293 ± ·1075	4·2959 ± ·0760	4·3076 ± ·0764	·8061 ± ·0088
	T_4 L		99·4290 ± ·1121	4·4767 ± ·0792	4·5024 ± ·0799	·8039 ± ·0089
	T_5 R	876	45·9612 ± ·0627	2·7499 ± ·0443	5·9831 ± ·0967	·6729 ± ·0125
	T_5 L		46·0000 ± ·0630	2·7664 ± ·0446	6·0096 ± ·0972	·6716 ± ·0125
	T_6 R	827	45·5266 ± ·0889	3·7915 ± ·0629	8·3281 ± ·1391	·7748 ± ·0094
	T_6 L		45·8761 ± ·0886	3·7788 ± ·0627	8·2370 ± ·1375	·7718 ± ·0095
	T_7 R	840	36·1359 ± ·0684	2·9405 ± ·0484	8·1374 ± ·1348	·7937 ± ·0086
	T_7 L		35·9714 ± ·0665	2·8565 ± ·0470	7·9410 ± ·1315	·7925 ± ·0087
Sphenoidal	S_2 R	719	75·0542 ± ·0863	3·4328 ± ·0611	4·5738 ± ·0815	·8315 ± ·0078
	S_2 L		74·7774 ± ·0881	3·5017 ± ·0623	4·6828 ± ·0835	·8280 ± ·0079
	S_3 R	866	36·2992 ± ·0443	1·9344 ± ·0314	5·3289 ± ·0866	·7771 ± ·0091
	S_3 L		36·5860 ± ·0462	2·0146 ± ·0327	5·5065 ± ·0895	·7733 ± ·0092
	S_5 R	722	56·6454 ± ·0864	3·4406 ± ·0611	6·0739 ± ·1082	·7962 ± ·0092
	S_5 L		56·3102 ± ·0865	3·4481 ± ·0612	6·1234 ± ·1091	·7926 ± ·0093
	S_6 R	808	35·5364 ± ·0569	2·3965 ± ·0402	6·7438 ± ·1137	·8776 ± ·0055
	S_6 L		35·5015 ± ·0579	2·4399 ± ·0409	6·8727 ± ·1159	·8750 ± ·0056
Malar	Ml_1 R	817	49·3853 ± ·0734	3·1118 ± ·0519	6·3011 ± ·1056	·9219 ± ·0035
	Ml_1 L		49·9642 ± ·0757	3·2068 ± ·0535	6·4182 ± ·1076	·9177 ± ·0037
	Ml_2 R	718	59·4213 ± ·1102	4·3794 ± ·0780	7·3701 ± ·1319	·9399 ± ·0029
	Ml_2 L		59·5829 ± ·1146	4·5542 ± ·0811	7·6435 ± ·1368	·9378 ± ·0030
Maxillary	Mx_1 R	692	66·5954 ± ·1007	3·9276 ± ·0712	5·8978 ± ·1073	·9766 ± ·0012
	Mx_1 L		66·3685 ± ·1001	3·9045 ± ·0708	5·8830 ± ·1070	·9738 ± ·0013
	Mx_2 R	517	55·6924 ± ·0851	2·8695 ± ·0602	5·1524 ± ·1084	·9278 ± ·0041
	Mx_2 L		55·9226 ± ·0857	2·8899 ± ·0606	5·1677 ± ·1087	·9228 ± ·0043
	Mx_3 R	451	40·8182 ± ·0997	3·1396 ± ·0705	7·6917 ± ·1738	·9134 ± ·0053
	Mx_3 L		40·8803 ± ·1031	3·2453 ± ·0729	7·9385 ± ·1794	·9093 ± ·0055
	Mx_4 R	545	38·3780 ± ·0749	2·5924 ± ·0530	6·7549 ± ·1386	·8616 ± ·0074
	Mx_4 L		38·3541 ± ·0795	2·6823 ± ·0548	6·9936 ± ·1436	·8556 ± ·0076
Occipital	O_7 R	864	97·3113 ± ·1225	5·3384 ± ·0866	5·4859 ± ·0893	·7954 ± ·0084
	O_7 L		98·6551 ± ·1271	5·5363 ± ·0899	5·6118 ± ·0914	·7940 ± ·0085
	O_8 R	858	63·7943 ± ·0849	3·6869 ± ·0600	5·7793 ± ·0944	·5379 ± ·0164
	O_8 L		63·3409 ± ·0758	3·2930 ± ·0536	5·1988 ± ·0849	·5345 ± ·0165
	O_9 R	858	74·7284 ± ·0791	3·4344 ± ·0559	4·5959 ± ·0750	·7928 ± ·0086
	O_9 L		74·4242 ± ·0760	3·2980 ± ·0537	4·4313 ± ·0723	·7892 ± ·0087

* The upper of the two correlations is found by the direct product moment method, the lower by the formula based on the three standard deviations.

Measurements in millimetres.

Constants of the Differences of Means			Constants of the Differences of Absolute Variabilities			Constants of the Differences of Relative Variabilities		
Mean Differences Δ_{R-L}	Standard Deviations σ_Δ	Ratio Δ/(p.e. of Δ)	Δ $\sigma_R - \sigma_L$	Standard Deviation of $\sigma_R - \sigma_L$	Ratio Δ/(p.e. of Δ)	V $V_R - V_L$	Standard Deviation of $V_R - V_L$	Ratio Δ/(p.e. of Δ)
+ ·3955 ± ·05036	2·2216 ± ·03561	+ 7·85	+ ·0421	·07228	+0·86	+ ·0220	·073,669	+0·44
+ ·5704 ± ·03959	1·7478 ± ·02798	+14·41	− ·1964	·05678	−5·13	− ·2601	·064,928	−5·94
+1·3302 ± ·11572	4·7177 ± ·08204	+11·49	+ ·2305	·15662	+2·18	+ ·1443	·140,115	+1·53
+2·2348 ± ·10402	4·5562 ± ·07354	+21·49	− ·1251	·14230	− 1·30	− ·1257	·086,747	−2·15
+1·4106 ± ·09508	3·8292 ± ·06724	+14·84	− ·1171	·13300	−1·31	− ·0932	·075,484	−1·83
+ ·8915 ± ·05453	2·3791 ± ·03858	+16·35	+ ·0580	·07641	+1·13	+ ·0210	·088,438	+0·35
− ·1694 ± ·05302	2·2489 ± ·03666	− 3·20	+ ·0723	·07425	+1·44	+ ·1838	·159,601	+1·71
+ ·4833 ± ·05369	2·3752 ± ·03838	+ 9·00	+ ·0163	·07600	+0·32	− ·0227	·115,443	−0·29
+ ·3003 ± ·06889	2·7524 ± ·04872	+ 4·36	− ·1808	·09643	−2·78	− ·1948	·096,973	−2·98
− ·0388 ± ·05095	2·2326 ± ·03597	− 0·76	− ·0165	·06891	−0·36	− ·0265	·150,184	−0·26
− ·3495 ± ·05996	2·5570 ± ·04240	− 5·83	+ ·0127	·08320	+0·23	+ ·0911	·182,488	+0·74
+ ·1645 ± ·04349	1·8690 ± ·03076	+ 3·78	+ ·0840	·06088	+2·05	+ ·1964	·169,394	+1·72
+ ·2768 ± ·05117	2·0345 ± ·03619	+ 5·41	− ·0689	·07186	−1·42	− ·1090	·096,031	−1·68
− ·2868 ± ·03052	1·3317 ± ·02159	− 9·40	− ·0802	·04227	−2·81	− ·1775	·116,113	−2·27
+ ·3352 ± ·05568	2·2183 ± ·03937	+ 6·02	− ·0075	·07755	−0·14	− ·0495	·137,625	−0·53
+ ·0349 ± ·02871	1·2098 ± ·02030	+ 1·22	− ·0434	·04080	−1·58	− ·1289	·115,154	−1·66
− ·5789 ± ·03033	1·2851 ± ·02145	−19·09	− ·0950	·04289	−3·28	− ·1171	·086,439	−2·01
− ·1616 ± ·03989	1·6497 ± ·02936	− 4·05	− ·1748	·05710	−4·54	− ·2734	·096,210	−4·21
+ ·2269 ± ·02298	·8961 ± ·01625	+ 9·87	+ ·0231	·03202	+1·07	+ ·0148	·048,190	+0·46
− ·2302 ± ·03358	1·1320 ± ·02375	− 6·86	− ·0204	·04727	−0·64	− ·0153	·084,790	−0·27
− ·0621 ± ·04331	1·3637 ± ·03063	− 1·43	− ·1057	·06129	−2·56	− ·2468	·150,509	−2·43
+ ·0239 ± ·04103	1·3006 ± ·02443	+ 0·58	− ·0899	·05741	−2·32	− ·2387	·150,022	−2·36
−1·3438 ± ·08022	3·4955 ± ·05673	−16·75	− ·1979	·11223	−2·61	− ·1259	·114,654	−1·63
+ ·4534 ± ·07796	3·3850 ± ·05511	+ 5·82	+ ·3939	·10071	+5·80	+ ·5805	·158,784	+5·42
+ ·3042 ± ·05043	2·1897 ± ·03565	+ 6·03	+ ·1364	·07010	+2·89	+ ·1646	·094,086	+2·59

It will be seen that of the 25 characters we can only say of four that no definite asymmetry is indicated. Further, of these four measurements none is of first-class importance so far as the brain is concerned, that of most interest being the chord from asterion to auricular point. A noteworthy fact is that none of the measurements gives us differences of right and left measurements lying between two and three times their probable errors—the region in which significance is doubtful. No less than 14 of the measurements fall into the markedly significant group, while another seven are with high probability significant. We conclude therefore that *the human skull from its very nature* (*like the internal organs of the human body*) *is asymmetrical*; it is not a question of asymmetry in the individual, but of asymmetry in the type. The sculptor who desires to form not a portrait, but a typical representative of man (or of a god in the image of man) must model the head *asymmetrically* *. The leading feature of this asymmetry is the predominance of the *right-hand* side. Examining Table II we find that the right side bones are predominant in 16 of the 25 measurements, as against nine on the left side. The table indicates further that the average measure of significance is 8·66 on the right as against 7·49 only on the left. Of the eight most significant differences, six are on the right side, only two on the left. *All* the measurements of the frontal and parietal bones show marked excess on the right side. They thus confirm the conclusion already reached in this journal† that the right cerebral hemisphere is the larger. Even with the sphenoid bone three out of the four measurements are predominant on the right, but the fourth measurement, the distance from the postreme point of the sphenoid to the spheno-basion, is markedly significant, and predominant on the left. The malar bone, so far as we can judge from two measurements, is predominant on the left side. The marked right predominance of the fundamental vertical measurement (Mx_1) of the maxillary bone possibly accounts for the nasal wryness which is so common, i.e. the slight drawing up of the nasal ala or even the mouth on the left. On the other hand, the horizontal Mx_2 is larger on the left, indicating that the left upper jaw is larger than the right. It is therefore possible that a correlated predominance of the left side of the mandible exists, and this point would be worth investigating.

Of the seven measurements of the temporal bone, one difference is practically of no significance, the distance from asterion to auricular point, T_5 being practically symmetrical.

Of the remaining six measurements four are predominant on the right side and two of these very markedly so. The two measurements predominant on the left—both vertical measurements—are significant but neither very markedly so. On the whole the temporal bone while not entirely dominant on the right side must be considered as part of that system of frontal and parietal bones which gives pre-eminence to the right side.

* Quite recently an obtuse writer laboriously measured the heads of Greek statues, and accused the sculptors of the Periclean age of making their gods asymmetrical!

† Hoadley and Pearson: "On Measurement of the Internal Diameters of the Skull." *Biometrika*, Vol. xxi. pp. 85—123.

TABLE II. *Dominance of Right and Left Cranial Bones, estimated by Average Size.*

	Right Dominance			Left Dominance		
	Bone	Length	$\frac{\Delta_{R-L}}{\text{p.e. of } \Delta_{R-L}}$	Bone	Length	$\frac{\Delta_{R-L}}{\text{p.e. of } \Delta_{R-L}}$
Markedly significant	Parietal	P_3	+21·49	—	—	—
	—	—	—	Malar	Ml_1	−19·09
	—	—	—	Occipital	O_7	−16·75
	Temporal	T_1	+16·35	—	—	—
	Parietal	P_4	+14·84	—	—	—
	Frontal	F_2	+14·41	—	—	—
	Parietal	P_2	+11·49	—	—	—
	Maxillary	Mx_1	+ 9·87	—	—	—
	—	—	—	Sphenoid	S_3	−9·40
	Temporal	T_2	+ 9·00	—	—	—
	Frontal	F_1	+ 7·85	—	—	—
	—	—	—	Maxillary	Mx_2	−6·86
	Occipital	O_9	+ 6·03	—	—	—
	Sphenoid	S_5	+ 6·02	—	—	—
Significant	—	—	—	Temporal	T_6	−5·83
	Occipital	O_8	+ 5·82	—	—	—
	Sphenoid	S_2	+ 5·41	—	—	—
	Temporal	T_4	+ 4·36	—	—	—
Probably significant	—	—	—	Malar	Ml_2	−4·05
	Temporal	T_7	+ 3·78	—	—	—
	—	—	—	Temporal	T_2	−3·20
Non-significant	—	—	—	Maxillary	Mx_3	−1·43
	Sphenoid	S_6	+ 1·22	—	—	—
	Maxillary	Mx_4	+ 0·58	—	—	—
	—	—	—	Temporal	T_5	−0·76
		Mean Right	+8·66		Mean Left	−7·49

Now it is somewhat difficult to realise how predominance of one side can arise without a counterbalancing predominance somewhere else on the other. We might possibly anticipate a greater predominance of the left side on the cerebellar and basal portions of the skull. The occipital arc, from lambda to asterion, O_7, is very significantly greater on the left, but not so the lower arc, O_8, nor the chord from basion to asterion, O_9. S_3 is again, however, greater on the left. It is clear that we cannot state any rule as to left predominance compensating for right predominance owing to their balancing on the skull. The asymmetries of the cranial bones do not equalise each other, so as to produce a symmetrical total head form, rather they tend to give a distorted form to the skull as a whole.

We can examine the problem from another standpoint, that of the percentage of cases on either side in which the right or left measurement is in excess. The

two methods, that of predominance of mean size, and that of predominance in number of individuals, need not necessarily lead to the same results. The reduced data will be found in Table III. It is needful, however, to consider first what is the probable error of the difference of two percentages in a population. Let the numbers corresponding to the two percentages p_s and p_t be n_s and n_t in a population of size N; then if p_{s-t} be the percentage difference

$$p_s = 100n_s/N, \quad p_t = 100n_t/N,$$

$$p_{s-t} = \frac{100}{N}(n_s - n_t) \text{ and } \overline{p}_{s-t} = \frac{100}{N}(\overline{n}_s - \overline{n}_t);$$

thus
$$\delta p_{s-t} = \frac{100}{N}(\delta n_s - \delta n_t),$$

$$\sigma^2_{p_{s-t}} = \left(\frac{100}{N}\right)^2 (\sigma^2_{n_s} + \sigma^2_{n_t} - 2\sigma_{n_s}\sigma_{n_t} r_{n_s n_t}).$$

But
$$\sigma^2_{n_s} = \tilde{n}_s\left(1 - \frac{\tilde{n}_s}{N}\right), \quad \sigma^2_{n_t} = \tilde{n}_t\left(1 - \frac{\tilde{n}_t}{N}\right),$$

and
$$\sigma_{n_s}\sigma_{n_t} r_{n_s n_t} = -\frac{\tilde{n}_s \tilde{n}_t}{N},$$

where $\tilde{n}_s$, $\tilde{n}_t$ are the reduced parent population values which for want of better information we put equal to the sample values. Thus

$$\sigma^2_{p_{s-t}} = \left(\frac{100}{N}\right)^2 \left(\tilde{n}_s + \tilde{n}_t - \frac{(\tilde{n}_s - \tilde{n}_t)^2}{N}\right)$$

$$= \frac{100}{N}\left(\tilde{p}_s + \tilde{p}_t - \frac{1}{100}(\tilde{p}_s - \tilde{p}_t)^2\right).$$

$$\text{Probable error of } p_{s-t} = \frac{6{\cdot}7449}{\sqrt{N}}\left(p_s + p_t - \frac{1}{100}(p_s - p_t)^2\right)^{\frac{1}{2}}$$

approximately, substituting the observed values.

We can now form Table IV (p. 336), corresponding to Table II, and arranged according to the significance of percentage differences. There is not much change in the order or magnitude of the significance of the several cranial lengths, whether we judge dominance by average size or relative percentage of excess. The tendency when using percentage excess of size is to somewhat reduce the position of the measurements. Taking, however, the "significant" and "markedly significant" differences 17 out of 18 remain in the same group; only the sphenoidal length S_2 has dropped out of the "significant" into the "probably significant" category. Mx_3 has passed from "non-significant" dominance on the left to the same category on the right; no other measurement has changed its dominance. In other words, whether we judge by percentage of excess in size, or by mean size, the bones on the right side of the skull possess a dominance in the ratio of about 12 to 5 in the classes where significance may be taken to be certain. Why the antero-posterior lengths of malar and occipital bones should be so markedly greater on the left, I am unable

TABLE III. *Percentages and Significance of their Differences.*

Character and number		Number: Greater on Right	Number: Equal*	Number: Greater on Left	Δ_P percentage difference	Probable error of Δ_P	Ratio
F_1 (885)	Nos.	404	210	271	+15·03	1·9503	+ 7·71
	%	45·65 ± 1·13	23·73 ± ·96	30·62 ± 1·04			
F_2 (887)	Nos.	475	192	220	+28·75	1·8963	+15·16
	%	53·55 ± 1·13	21·65 ± ·93	24·80 ± ·98			
P_2 (754)	Nos.	443	61	250	+25·60	2·2691	+11·28
	%	58·75 ± 1·21	8·09 ± ·67	33·16 ± 1·16			
P_3 (873)	Nos.	561	89	223	+38·72	1·9976	+19·38
	%	64·26 ± 1·09	10·19 ± ·69	25·55 ± 1·00			
P_4 (738)	Nos.	439	80	219	+29·82	2·2245	+13·40
	%	59·49 ± 1·22	10·84 ± ·77	29·67 ± 1·13			
T_1 (866)	Nos.	499	135	232	+30·83	1·9837	+15·54
	%	57·62 ± 1·13	15·59 ± ·83	26·79 ± 1·02			
T_2 (856)	Nos.	321	147	388	− 7·83	2·0899	− 3·75
	%	37·50 ± 1·12	17·17 ± ·87	45·33 ± 1·15			
T_3 (871)	Nos.	423	173	275	+16·99	2·0083	+ 8·46
	%	48·57 ± 1·14	19·86 ± ·91	31·57 ± 1·06			
T_4 (726)	Nos.	339	121	266	+10·06	2·2710	+ 4·43
	%	46·69 ± 1·25	16·67 ± ·93	36·64 ± 1·21			
T_5 (876)	Nos.	345	175·5	355·5	− 1·20	2·0378	− 0·59
	%	39·38 ± 1·11	20·04 ± ·91	40·58 ± 1·12			
T_6 (827)	Nos.	293	144	390	−11·73	2·1133	− 5·55
	%	35·43 ± 1·12	17·41 ± ·89	47·16 ± 1·17			
T_7 (840)	Nos.	341	203	296	+ 5·36	2·0226	+ 2·65
	%	40·60 ± 1·14	24·16 ± 1·00	35·24 ± 1·11			
S_2 (719)	Nos.	293	194	232	+ 8·48	2·1385	+ 3·97
	%	40·75 ± 1·24	26·98 ± 1·12	32·27 ± 1·18			
S_3 (866)	Nos.	237	268	361	−14·28	1·8767	− 7·61
	%	27·40 ± 1·02	20·91 ± 1·06	41·69 ± 1·13			
S_5 (722)	Nos.	338	138	246	+12·74	2·2347	+ 5·70
	%	46·82 ± 1·25	19·11 ± ·99	34·07 ± 1·19			
S_6 (808)	Nos.	262	287	259	+ 0·37	1·9055	+ 0·20
	%	32·43 ± 1·11	35·52 ± 1·14	32·05 ± 1·11			
Ml_1 (817)	Nos.	153	255	409	−31·33	1·8123	−17·29
	%	18·73 ± ·92	31·21 ± 1·09	50·06 ± 1·18			
Ml_2 (718)	Nos.	239	197	282	− 5·99	2·1388	− 2·80
	%	33·29 ± 1·19	27·44 ± 1·12	39·27 ± 1·23			
Mx_1 (692)	Nos.	246	315	131	+16·62	1·8439	+ 9·01
	%	35·55 ± 1·23	45·52 ± 1·28	18·93 ± 1·00			
Mx_2 (517)	Nos.	124	197	196	−13·93	2·2966	− 6·06
	%	23·98 ± 1·27	38·11 ± 1·44	37·91 ± 1·44			
Mx_3 (451)	Nos.	149	156	146	+ 0·67	1·8711	+ 0·36
	%	33·04 ± 1·49	34·59 ± 1·51	32·37 ± 1·49			
Mx_4 (545)	Nos.	193	175	177	+ 2·94	2·3789	+ 1·23
	%	35·41 ± 1·38	32·11 ± 1·35	32·48 ± 1·39			
O_7 (864)	Nos.	262	77	525	−30·44	2·0761	−14·66
	%	30·33 ± 1·06	8·91 ± ·65	60·76 ± 1·12			
O_8 (858)	Nos.	410	125	323	+10·14	2·1158	+ 4·79
	%	47·79 ± 1·15	14·57 ± ·81	37·64 ± 1·12			
O_9 (858)	Nos.	388	166	304	+ 9·79	2·0559	+ 4·76
	%	45·22 ± 1·15	19·35 ± ·91	35·43 ± 1·10			

* These are the percentages of the characters equal, not to the unit of measurement, but to the unit of grouping used in the correlation tables. The grouping unit was 1 mm. in 21 cases, 0·6 mm. in 2 cases (S_3 and S_6) and 0·5 mm. in 2 cases (T_5 and T_7).

TABLE IV. *Dominance of Right and Left Cranial Bones, estimated by Percentage Excess.*

	Right Percentage Excess			Left Percentage Excess		
	Bone	Length	$\frac{\Delta_{p_R - p_L}}{\text{p.e. of } \Delta}$	Bone	Length	$\frac{\Delta_{p_R - p_L}}{\text{p.e. of } \Delta}$
Markedly significant	Parietal	P_3	+19·38	—	—	—
	—	—	—	Malar	Ml_1	−17·29
	Temporal	T_1	+15·54	—	—	—
	Frontal	F_2	+15·16	—	—	—
	—	—	—	Occipital	O_7	−14·66
	Parietal	P_4	+13·40	—	—	—
	Parietal	P_2	+11·28	—	—	—
	Maxillary	Mx_1	+ 9·01	—	—	—
	Temporal	T_3	+ 8·46	—	—	—
	Frontal	F_1	+ 7·71	—	—	—
	—	—	—	Sphenoid	S_3	− 7·61
	—	—	—	Maxillary	Mx_2	− 6·06
Significant	Sphenoid	S_5	+ 5·70	—	—	—
	—	—	—	Temporal	T_6	− 5·55
	Occipital	O_8	+ 4·79	—	—	—
	Occipital	O_9	+ 4·76	—	—	—
	Temporal	T_4	+ 4·43	—	—	—
Probably significant	Sphenoid	S_2	+ 3·97	—	—	—
	—	—	—	Temporal	T_2	− 3·75
	—	—	—	Malar	Ml_2	− 2·80
	Temporal	T_7	+ 2·65	—	—	—
Non-significant	Maxillary	Mx_4	+ 1·23	—	—	—
	—	—	—	Temporal	T_5	− 0·59
	Maxillary	Mx_3	+ 0·36	—	—	—
	—	S_6	+ 0·20	—	—	—
	Mean Dominance Right		+ 7·54	Mean Dominance Left		− 7·29

to say. The value of $Ml_1 + T_4 + O_7$, which is very nearly the whole arc from sub-orbital point to lambda, via the asterion, is 1·62 mm. greater on the left than on the right side, while the parietal arc, P_4, from lambda to sphenion, is greater by 1·41 mm. on the right; this would seem to indicate that the dominance on the left side, if due at all to brain growth, is cerebellar, as the malar length can be less influenced by such growth.

It is one thing for homologous lengths to differ in mean size, another for homologous lengths to be highly correlated, which signifies that their deviations from their respective means are closely related. It will be now of interest to arrange the twenty-five pairs of homologous lengths in their order of correlation. This is done in Table V.

The main feature of this table is that the facial lengths are those most highly correlated, while those of the temporal, occipital and parietal regions are less closely associated. Suppose a type skull formed with the average asymmetries we have shown to exist, then if any individual skull deviated from these type asymmetries on the right side, there would be a correlated deviation in the same sense on the left side, and these deviations would be in closer accordance on the anterior or facial portion of the cranium than on parts posterior to the coronal suture.

TABLE V. *Correlations of Homologous Lengths in order of Intensity.*

Length	Correlation	Length	Correlation	Length	Correlation
Mx_1	·9766	T_3	·8503	P_4	·7909
Ml_2	·9399	S_2	·8315	S_3	·7771
F_2	·9384	T_2	·8145	T_6	·7748
Mx_2	·9278	T_1	·8133	P_3	·7112
Ml_1	·9219	T_4	·8061	P_2	·6738
Mx_3	·9134	S_5	·7962	T_5	·6729
F_1	·9059	O_7	·7954	O_8	·5379
S_6	·8776	T_7	·7937	—	—
Mx_4	·8616	O_9	·7928	—	—

(5) *Variation.*

If the right side of the cranium is on the whole significantly dominant in size, it remains to consider the distribution of the variability, absolute and relative, of the skull. Is the right or the left side the more subject to limitation in its variation; is either by reason of its functions more stringently bound to type than its opposite? Or, shall we find equality of variability in homologous lengths notwithstanding their divergence in size?

The data for answering this problem are provided by the last six columns of Table I. In the first three of these columns absolute variabilities are dealt with. We have the difference of the standard deviations on the right and the left, then the standard error of this difference, which is provided by the formula*

$$\sigma_{\sigma_R - \sigma_L} = \frac{1}{\sqrt{2n}} \sqrt{\sigma_R^2 + \sigma_L^2 - 2r_{RL}^2 \sigma_R \sigma_L},$$

and in the third column we have the ratio of the difference of the standard deviations to its probable error (i.e. ·67449 × standard error). In the last three columns relative variabilities are dealt with. In the first we have the difference of the coefficients of variation for the right and left homologous lengths; in the second of these three columns we have the standard error of the difference of these coefficients provided by the formula*

$$\sigma_{V_R - V_L} = \frac{1}{\sqrt{2n}} \left\{ V_R^2 + V_L^2 - 2r^2 V_R V_L + \frac{2}{(100)^2} (V_R^4 + V_L^4 - 2r V_R^2 V_L^2) \right\}^{\frac{1}{2}},$$

* Here r is the correlation of the right and left homologous lengths. Of course these formulae are only approximations suitable to large samples, such as they are in our case.

and the last column gives the ratio of the difference of the coefficients of variation to the probable error of that difference.

From the ratio columns, Table VI has been drawn up. We can draw at once certain conclusions from Table VI. It will be seen that the cases of marked significance, which were so noteworthy when we considered dominance of size, do not occur at all. In other words, laterality is not a marked feature of variability either relative or absolute. Again, while the size dominance was on the *right* in the proportion of 16 to 9, the dominance of variability is on the *left* in the

TABLE VI. *Significance of the differences of Relative and Absolute Variability on the two sides of the skull.*

	Absolute Variability				Relative Variability			
	Bone	Length	Ratio		Bone	Length	Ratio	
			Right in excess	Left in excess			Right in excess	Left in excess
Significant	Occipital	O_8	+5·80	—	Frontal	F_2	—	−5·94
	Frontal	F_2	—	−5·13	Occipital	O_8	+5·42	—
	Malar	Ml_2	—	−4·54	Malar	Ml_2	—	−4·21
Possibly significant	Malar	Ml_1	—	−3·28	Temporal	T_4	—	−2·98
	Occipital	O_9	+2·89	—	Occipital	O_9	+2·59	—
	Sphenoid	S_3	—	−2·81	Maxillary	Mx_3	—	−2·43
	Temporal	T_4	—	−2·78	—	—	—	—
	Occipital	O_7	—	−2·61	—	—	—	—
	Maxillary	Mx_3	—	−2·56	—	—	—	—
Very doubtful significance	Maxillary	Mx_4	—	−2·32	Maxillary	Mx_4	—	−2·36
	Parietal	P_2	+2·18	—	Sphenoid	S_3	—	−2·27
	Temporal	T_7	+2·05	—	Parietal	P_3	—	−2·15
	—	—	—	—	Malar	Ml_1	—	−2·01
Non-significant	Sphenoid	S_6	—	−1·58	Parietal	P_4	—	−1·83
	Temporal	T_2	+1·44	—	Temporal	T_7	+1·72	—
	Sphenoid	S_2	—	−1·42	Temporal	T_2	+1·71	—
	Parietal	P_4	—	−1·31	Sphenoid	S_2	—	−1·68
	Parietal	P_3	—	−1·30	Sphenoid	S_6	—	−1·66
	Temporal	T_1	+1·13	—	Occipital	O_7	—	−1·63
	Maxillary	Mx_1	+1·07	—	Parietal	P_2	+1·53	—
	Frontal	F_1	+0·86	—	Temporal	T_6	+0·74	—
	Maxillary	Mx_2	—	−0·64	Sphenoid	S_5	—	−0·53
	Temporal	T_5	—	−0·36	Maxillary	Mx_1	+0·46	—
	Temporal	T_3	+0·32	—	Frontal	F_1	+0·44	—
	Temporal	T_6	+0·23	—	Temporal	T_1	+0·35	—
	Sphenoid	S_5	—	−0·14	Temporal	T_3	—	−0·29
	—	—	—	—	Maxillary	Mx_2	—	−0·27
	—	—	—	—	Temporal	T_5	—	−0·26

proportion of 15 to 10. Out of those cases for which dominance is either significant or possibly significant the ratio in *favour* of the *left side* is 7 to 2 for absolute and 8 to 2 for relative variability, giving a 15 to 4 proportion instead of 15 to 10. There seems little doubt therefore that the left side is somewhat less limited to type than the right side of the skull*. Confining our attention to the really significant group we remark that the chance of equalling or exceeding $\pm 4{\cdot}21$ times the probable error in one trial is only about $\frac{1}{250}$, and therefore the probability that in 50 trials we should get six such values is exceedingly small.

We see that the undoubtedly significant group consists solely of three lengths, one from the occipital with right dominance, two with left dominance from the frontal and malar bones. With one exception, T_3, the lengths with dominance for absolute variability have the same laterality for relative variability, so that we need not distinguish between the two. Of the 25 characters the dominances in size and absolute variability have the same laterality in 15 cases, the opposite laterality in 10 cases. Of these 10 cases (judged by variability) the difference is markedly significant in one, F_2, possibly significant in two, T_4 and Mx_4, and non-significant in seven. In Mx_4 the size difference is non-significant, but it is significant in F_2 and T_4. Of the 15 cases in which the dominance in size and in variability has the same laterality, O_8 has significance for both, the malar bone measurement, Ml_2, has significance for both, Ml_1 has marked significance for size, and doubtful significance for variability; of the six quantities which are possibly or just possibly significant for variability, P_2, S_3, O_7 and O_9 are markedly significant for size, T_7 is probably significant for size and Mx_3 is non-significant for size. There are six cases in which the variability dominance has no significance. Thus in the case where the dominances are of unlike sense there are only two measurements, F_2 and T_4, in which it is almost certainly significant for both size and variability. In the case where the dominance is of like sense, the lengths O_8 and Ml_2 have adequate significance for both size and variability; O_7, O_9, S_3 and Ml_1 have doubtful significance for one or other character, and P_2, T_7 have extremely doubtful significance for variability. Accordingly we have left four lengths distributed over four bones, F_2, T_4, Ml_2 and O_8, two of which have unlike and two like dominance in size and variability. Thus it seems idle to argue from these as to any correlation, positive or negative, existing between dominance in size and in variability†. It is clear that laterality has far less influence on variability than it has on size, and less on relative variability than on absolute variability.

(6) The conclusions of this paper are of the following kind:

(i) The human skull is definitely and markedly asymmetrical. It is not a question of the bones of individual crania differing from a symmetrical type, but the type cranium is itself asymmetrical.

* The odds against such an excess of dominance on the left are about 29 to 1.

† The ratios of significance for size and for absolute variability were correlated and gave the result $0{\cdot}2939 \pm {\cdot}1232$. Thus significant greater variability was associated with significant greater size, and not, as one might *a priori* suppose, a stringent predominance of size with a lesser variability. But the correlation is under 2·5 times its probable error, and it is too doubtful in itself for one to say more than that there is not sufficient evidence to indicate a relation between dominance in size and variability.

(ii) Some dimensions of the cranial bones have dominance on the right side, some on the left, but on the whole the right side for size has dominance over the left. This is especially true for the frontal and parietal bones; the malar bone is the only case in which the left side has dominance for all measurements taken, and this bone has less relation to brain development.

(iii) The anterior homologous lengths, particularly those of the face and forehead, are those most highly correlated, right and left.

(iv) The order of absolute variability is much the same as that of relative variability. There are no cases of markedly significant differences in variability of right and left bones. There are only three cases of definitely significant differences of variability, one on the right and two on the left. No relation of any importance was discovered between dominance in size and dominance in variability.

(v) Whatever causes, associated with brain growth, or otherwise, lead to dominance in size of certain lateral portions of the skull, these do not appear to restrict the variability of those portions in any sensible degree. That is to say, type is differentiated laterally, but not deviations from type.

Frontal Bone, F_1. (Central Values.) Left.

Frontal Bone, F_2. (Central Values.) Left.

TABLE VII.

Frontal Arc Measurement F_1.

Frontal Bone, F_1. (Central Values.) Right.

Frontal Bone, F_1. (Central Values.) Left.

	82·75	83·75	84·75	85·75	86·75	87·75	88·75	89·75	90·75	91·75	92·75	93·75	94·75	95·75	96·75	97·75	98·75	99·75	100·75	101·75	102·75	103·75	104·75	105·75	106·75	107·75	108·75	109·75	110·75	111·75	112·75	113·75	114·75	115·75	116·75	117·75	118·75	
80·75	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
81·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
82·75	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
83·75	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
84·75	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
85·75	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
86·75	—	—	—	—	1	2	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
87·75	—	—	—	—	—	1	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
88·75	—	—	—	1	—	1	5	3	4	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	16
89·75	—	—	1	—	1	2	1	5	4	4	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	21
90·75	—	—	—	—	—	—	—	2	5	6	3	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	20
91·75	—	—	—	—	—	—	—	1	2	4	5	5	5	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	26
92·75	—	—	—	—	—	—	—	2	3	7	9	8	4	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	35
93·75	—	—	—	—	—	—	—	1	3	4	7	14	13	4	4	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	53
94·75	—	—	—	—	—	1	—	1	—	2	3	7	11	10	9	9	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	57
95·75	—	—	—	—	—	—	—	—	1	2	1	6	7	22	18	9	5	5	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	80
96·75	—	—	—	—	—	—	—	—	—	—	—	3	6	8	25	13	10	5	4	3	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	83
97·75	—	—	—	—	—	—	—	—	—	—	1	2	3	9	18	22	8	8	7	5	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	85
98·75	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2	4	9	14	10	5	3	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	53
99·75	—	—	—	—	—	—	—	—	—	—	—	1	1	1	2	3	7	11	12	4	4	4	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	52
100·75	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2	3	6	6	12	7	5	2	3	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	51
101·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	3	5	12	15	7	6	3	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	53
102·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	2	7	10	13	6	4	3	3	1	—	1	—	—	—	—	—	—	—	—	—	—	52
103·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2	4	6	6	7	7	3	—	1	—	1	—	—	—	—	—	—	—	—	—	—	38
104·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2	1	2	3	4	3	3	1	—	—	—	—	—	—	—	—	—	—	—	—	20
105·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	—	1	2	3	3	3	4	4	—	—	—	—	—	—	—	—	—	—	—	—	23
106·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	3	3	4	2	1	1	1	—	—	—	—	—	—	—	—	—	17
107·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	1	1	2	3	2	1	—	1	—	—	—	—	—	—	—	—	13
108·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	—	1	3	—	—	—	—	—	—	—	—	—	—	7
109·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	—	2	—	—	—	—	—	—	—	—	—	4
110·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	2	1	—	—	—	—	—	—	—	—	—	—	4
111·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	2
112·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1
113·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
114·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
115·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
116·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
117·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1
118·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1
	1	2	4	3	2	7	8	15	23	31	33	51	53	61	82	66	55	61	76	59	45	38	32	22	18	13	7	8	3	1	1	1	—	—	1	1	1	885

TABLE VIII.

Frontal Arc Measurement F_2.

Frontal Bone, F_2. (Central Values.) Right.

Frontal Bone, F_2. (Central Values.) Left.

	73·75	74·75	75·75	76·75	77·75	78·75	79·75	80·75	81·75	82·75	83·75	84·75	85·75	86·75	87·75	88·75	89·75	90·75	91·75	92·75	93·75	94·75	95·75	96·75	97·75	98·75	99·75	100·75	101·75	102·75	103·75	104·75	105·75	106·75	
72·75	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
73·75	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
74·75	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
75·75	—	—	—	—	1	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
76·75	—	—	2	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
77·75	—	—	—	1	1	2	3	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9
78·75	—	—	—	—	2	2	4	7	4	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	22
79·75	—	—	1	—	1	3	2	2	3	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	16
80·75	—	—	—	—	—	—	6	5	5	4	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	23
81·75	—	—	—	—	—	—	2	1	11	11	13	7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	45
82·75	—	—	—	—	—	—	1	—	4	9	10	18	8	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	52
83·75	—	—	—	—	—	—	2	—	3	6	8	11	13	6	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	52
84·75	—	—	—	—	—	—	—	—	1	5	6	15	16	14	9	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	63
85·75	—	—	—	—	—	—	—	—	2	5	12	12	21	15	5	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	76
86·75	—	—	—	—	—	—	—	—	—	1	3	4	8	14	18	8	7	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	65
87·75	—	—	—	—	—	—	—	—	—	—	—	1	4	8	16	11	12	10	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	63
88·75	—	—	—	—	—	—	—	—	—	—	—	3	3	3	6	23	21	9	3	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	73
89·75	—	—	—	—	—	—	—	—	—	—	—	1	—	2	6	8	22	11	10	8	—	—	—	—	—	—	—	—	—	—	—	—	—	—	68
90·75	—	—	—	—	—	—	—	—	—	—	—	—	1	1	4	11	11	9	9	11	8	—	—	—	—	—	—	—	—	—	—	—	—	—	65
91·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	4	6	5	5	5	—	—	—	—	—	—	—	—	—	—	—	—	—	33
92·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2	5	9	8	5	—	—	—	—	—	—	—	—	—	—	—	—	34
93·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	3	3	4	8	5	—	—	—	—	—	—	—	—	—	—	—	25
94·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	7	1	6	7	4	—	—	—	—	—	—	—	—	—	—	33
95·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	1	1	4	3	—	—	—	—	—	—	—	—	—	18
96·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	4	5	3	1	—	—	—	—	—	—	15
97·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	—	1	—	2	—	1	—	—	—	—	—	7
98·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	2	1	—	—	1	—	—	—	—	5
99·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	1	—	1	—	—	—	—	—	4
100·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	1
101·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1	—	2
102·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	—	2
103·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	1	2	3	4	7	9	17	18	34	42	50	73	65	71	74	73	80	51	39	46	35	35	19	16	11	4	2	1	1	2	—	—	1	1	887

TABLE IX.

Parietal Arc Measurement P_2.

Parietal Bone, P_2. (Central Values.) Right.

Parietal Bone, P_1. (Central Values.) Left.

	90·75	91·75	92·75	93·75	94·75	95·75	96·75	97·75	98·75	99·75	100·75	101·75	102·75	103·75	104·75	105·75	106·75	107·75	108·75	109·75	110·75	111·75	112·75	113·75	114·75	115·75	116·75	117·75	118·75	119·75	120·75	121·75	122·75	123·75	124·75	125·75	126·75	127·75	128·75	
89·75	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
90·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
91·75	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
92·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
93·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
94·75	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
95·75	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
96·75	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
97·75	—	—	—	—	—	—	1	—	—	—	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
98·75	—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
99·75	—	—	—	—	—	—	—	—	—	—	—	—	—	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
100·75	—	—	—	—	—	—	—	—	1	—	1	—	—	—	1	1	1	2	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9
101·75	—	—	—	—	—	—	—	1	—	—	—	3	2	2	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	10
102·75	—	—	—	—	—	—	—	—	—	—	1	—	1	—	1	2	—	4	—	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	13
103·75	—	—	—	—	—	—	—	—	—	—	—	1	1	—	1	—	3	—	1	1	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	16
104·75	—	—	—	—	—	—	—	—	—	—	—	1	2	2	2	3	4	3	3	—	1	2	2	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	26
105·75	—	—	—	—	—	—	—	1	1	—	—	2	—	2	—	4	4	5	3	3	2	2	2	2	2	—	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	40
106·75	—	—	—	—	—	—	—	—	—	—	1	—	—	—	3	3	2	6	4	6	5	2	2	3	2	2	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	42
107·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2	5	8	7	9	—	3	2	2	2	2	1	1	—	—	—	—	—	—	1	—	1	—	—	—	51
108·75	—	—	—	—	—	—	—	1	—	—	—	1	1	—	1	1	4	6	5	8	7	6	2	3	3	4	3	3	3	3	1	1	1	1	—	—	—	—	—	—	—	55
109·75	—	—	—	—	—	—	—	—	—	—	1	—	—	2	1	2	2	5	8	4	3	6	3	5	1	8	2	—	2	—	—	—	1	—	1	—	—	—	—	—	—	57
110·75	—	—	—	—	—	—	—	—	1	—	—	1	—	—	2	2	1	4	4	3	—	5	4	6	6	8	7	1	—	1	—	1	—	1	1	—	—	—	—	—	—	59
111·75	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	2	2	1	—	2	3	6	2	5	6	6	4	2	1	1	1	—	1	3	—	—	—	—	—	—	—	48
112·75	—	—	—	—	—	—	—	—	—	—	1	—	2	1	3	1	2	—	2	6	2	3	4	5	2	1	4	3	—	1	—	1	—	—	—	—	—	—	—	—	—	39
113·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	1	2	1	1	6	4	5	5	1	2	6	3	—	—	3	—	—	1	—	—	—	—	—	—	—	43
114·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	3	2	5	1	1	2	1	4	4	5	6	2	2	5	—	—	—	—	—	—	—	—	—	—	47
115·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	3	1	1	1	2	5	6	3	5	2	5	—	—	—	—	—	—	1	—	—	—	—	40
116·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	2	2	1	2	6	2	4	3	2	2	—	—	1	1	1	—	—	—	—	—	29
117·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2	3	—	—	—	—	2	4	1	2	4	1	1	3	1	—	—	—	—	—	—	—	27
118·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	6	1	2	2	2	3	—	1	1	—	1	—	1	—	—	21
119·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2	—	—	—	—	1	1	2	1	1	1	2	—	—	1	—	—	1	—	—	17
120·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	1	—	1	1	1	2	1	1	1	1	—	—	—	1	2	—	1	2	—	1	—	15
121·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	1	2	—	—	—	—	2	—	1	—	—	8
122·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	1	1	1	—	2	—	1	1	—	—	8
123·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2	1	—	—	—	1	—	—	—	—	1	—	—	—	—	6
124·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	—	—	1	3
125·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	2
126·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	1
127·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	1
128·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
	1	—	—	—	1	1	4	5	1	4	10	9	12	25	32	36	46	46	44	46	46	36	44	46	58	44	38	22	21	17	16	10	13	7	5	3	2	1	2	754

Parietal Bone, P_3. (Central Values.) Left.

TABLE X.

Parietal Arc Measurement P_3.

Parietal Bone, P_3. (Central Values.) Right.

	149·75	150·75	151·75	152·75	153·75	154·75	155·75	156·75	157·75	158·75	159·75	160·75	161·75	162·75	163·75	164·75	165·75	166·75	167·75	168·75	169·75	170·75	171·75	172·75	173·75	174·75	175·75	176·75	177·75	178·75	179·75	180·75	181·75	182·75	183·75	184·75	185·75	
142·75	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
143·75	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
144·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
145·75	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
146·75	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
147·75	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
148·75	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
149·75	—	1	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
150·75	—	—	1	1	—	1	1	2	—	2	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9
151·75	—	2	1	1	2	1	1	—	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	—	—	11
152·75	—	—	—	1	—	1	2	3	4	1	3	1	—	—	—	—	1	—	—	—	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	12
153·75	—	—	2	1	1	1	2	2	1	2	1	2	3	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	19
154·75	—	—	—	—	1	2	1	1	3	3	3	3	2	—	1	4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	24
155·75	—	—	—	—	2	1	2	3	3	4	—	3	3	1	—	1	—	—	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	28
156·75	—	—	—	—	1	—	—	3	3	3	3	5	2	3	1	1	2	1	1	—	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	31
157·75	1	—	—	3	—	—	—	—	3	3	4	6	4	2	2	1	2	1	2	4	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	39
158·75	—	—	—	—	—	1	—	—	3	3	3	6	3	1	5	3	2	1	3	2	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	41
159·75	—	—	—	—	—	2	1	—	1	3	6	4	2	4	3	6	7	4	1	1	—	2	1	1	1	1	—	—	1	—	—	—	—	—	—	—	—	—	—	51
160·75	—	—	—	—	1	1	—	2	2	4	2	2	7	6	8	2	6	1	4	2	1	1	1	—	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	53
161·75	—	—	—	—	1	1	—	5	1	2	1	2	3	8	8	1	4	8	4	3	5	2	—	2	2	—	—	1	—	—	—	—	—	—	—	—	—	—	—	59
162·75	—	—	—	—	—	—	—	1	2	2	3	3	8	4	2	4	2	6	2	4	2	4	2	3	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	49
163·75	—	—	—	—	—	—	1	—	—	2	1	4	4	4	7	8	5	5	—	2	6	2	1	2	—	1	2	—	—	—	—	—	—	—	—	—	—	—	—	54
164·75	—	—	—	—	—	—	—	2	—	2	2	1	6	6	4	5	7	5	7	4	5	1	1	1	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	60
165·75	—	—	—	—	—	—	—	—	1	1	4	2	1	6	6	5	9	9	4	1	4	5	1	1	3	2	—	1	—	—	—	—	—	—	—	—	—	—	—	58
166·75	—	—	—	—	—	—	—	1	—	—	1	2	1	1	1	1	3	9	4	1	4	5	5	—	3	—	1	—	—	—	—	—	—	—	—	—	—	—	—	41
167·75	—	—	—	—	—	—	—	—	—	1	—	1	2	1	—	3	6	2	3	3	6	5	—	2	—	1	2	—	—	1	2	—	—	—	—	—	—	—	—	1	39
168·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	4	1	3	3	6	2	4	2	4	3	—	—	—	—	—	1	—	—	—	—	—	—	—	—	38
169·75	—	—	—	—	—	—	—	—	1	—	—	—	—	—	2	3	4	1	5	4	7	2	3	1	2	2	—	—	—	1	1	—	—	—	—	—	—	—	—	—	32
170·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	4	2	2	6	4	6	3	1	3	1	—	—	1	1	—	—	—	—	—	—	—	—	—	35
171·75	—	—	—	—	—	—	—	—	1	—	—	—	2	—	—	1	2	—	1	1	5	4	5	3	1	—	4	1	—	—	—	—	—	—	—	—	—	—	—	—	17
172·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	1	1	1	1	3	3	1	2	—	1	1	—	—	—	—	—	—	—	—	16
173·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	2	1	1	5	1	1	—	—	1	1	1	—	—	—	1	—	—	—	—	7
174·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	1	—	1	1	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	4
175·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	3
176·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—
177·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
178·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2
179·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	3
180·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1
181·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
182·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
183·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
184·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	1
	3	3	4	10	10	11	10	23	30	34	34	44	49	49	47	50	70	61	54	41	52	49	42	15	25	14	11	9	5	6	3	1	—	1	1	—	2	873

Temporal Bone, T_1. (Central Values.) Left.

Parietal Bone, P_4. (Central Values.) Left.

TABLE XI.

Parietal Arc Measurement P_4.

Parietal Bone, P_4. (Central Values.) Right.

Parietal Bone, P_4. (Central Values.) Left.

	150·75	157·75	158·75	159·75	160·75	161·75	162·75	163·75	164·75	165·75	166·75	167·75	168·75	169·75	170·75	171·75	172·75	173·75	174·75	175·75	176·75	177·75	178·75	179·75	180·75	181·75	182·75	183·75	184·75	185·75	186·75	187·75	188·75	189·75	190·75	
155·75	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
156·75	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
157·75	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
158·75	—	—	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
159·75	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
160·75	—	—	—	—	—	—	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
161·75	—	—	1	—	—	1	—	—	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
162·75	—	—	—	—	—	—	1	1	—	2	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
163·75	—	—	—	—	1	—	—	—	—	—	—	1	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
164·75	—	—	—	—	—	1	—	—	—	—	1	—	1	4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8
165·75	—	—	—	—	—	—	—	—	—	—	1	—	4	—	—	4	2	—	1	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	14
166·75	—	—	—	—	—	—	—	—	1	—	2	1	4	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	10
167·75	—	—	—	—	—	—	—	1	—	1	—	1	—	4	—	—	3	3	—	1	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	23
168·75	—	—	—	—	—	—	1	—	—	1	2	3	2	1	1	2	1	1	2	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	19
169·75	—	—	—	—	—	—	—	—	—	—	—	4	2	4	5	2	5	3	3	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	31
170·75	—	—	—	—	—	—	—	1	—	1	1	4	3	4	3	4	1	4	3	4	3	1	1	2	—	—	—	—	—	—	—	—	—	—	—	—	33
171·75	—	—	—	—	—	—	—	—	—	—	—	—	2	4	7	7	5	5	6	2	—	2	4	1	—	1	—	2	—	—	—	—	—	—	—	—	48
172·75	—	—	—	—	—	—	—	—	—	—	—	—	2	2	2	4	7	4	4	3	5	3	1	1	—	—	—	—	—	—	—	—	—	—	—	—	38
173·75	—	—	—	—	—	—	—	—	—	1	—	1	3	3	3	1	5	5	4	3	2	2	2	2	—	—	—	—	—	—	—	—	—	—	—	—	35
174·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	4	4	4	7	7	3	5	4	3	—	—	1	—	—	—	—	—	—	—	—	—	43
175·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	3	3	1	7	5	6	3	4	3	3	1	—	—	—	—	—	—	—	—	—	—	42
176·75	—	—	—	—	—	—	—	—	—	—	1	—	1	1	1	—	1	9	4	5	10	6	8	6	7	—	4	1	—	—	1	—	—	—	—	—	66
177·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	6	4	4	7	8	8	7	5	6	1	3	3	—	—	1	1	—	—	—	57
178·75	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	2	3	3	3	8	8	8	2	4	2	5	1	2	—	—	—	—	—	—	—	50
179·75	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	2	—	—	3	4	4	6	5	3	4	3	2	1	—	—	1	—	—	—	—	—	41
180·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	3	1	2	6	3	3	6	5	—	5	3	2	1	—	—	—	—	—	41
181·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	4	1	1	3	2	1	1	4	1	1	—	—	—	—	—	—	20
182·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	1	—	3	2	1	4	1	—	1	1	1	—	—	—	17
183·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	2	1	1	—	1	5	1	1	1	—	2	—	—	—	1	19
184·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	1	—	3	3	2	2	2	1	1	—	—	1	—	—	—	23
185·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	1	—	2	1	1	—	2	2	2	13
186·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	2	—	1	—	—	—	—	—	1	—	6
187·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1	1	2	—	—	—	—	1	—	—	—	6
188·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	1	1	3
189·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	1	1	3
190·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	1	—	1	3	3	2	2	2	4	4	9	15	11	30	31	30	35	42	51	51	44	53	57	55	36	42	20	27	26	12	9	10	4	4	3	3	2	4	738

TABLE XII.

Temporal Chord Measurement T_1.

Temporal Bone, T_1. (Central Values.) Right.

Temporal Bone, T_1. (Central Values.) Left.

	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	
74	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
75	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
76	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
77	—	—	1	—	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
78	—	—	—	2	2	—	1	1	3	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	11
79	—	—	—	2	4	—	1	4	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	16
80	—	—	—	—	4	—	2	3	4	2	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	20
81	—	—	—	1	1	—	6	5	11	7	5	1	—	—	—	—	—	—	—	—	—	—	—	—	—	37
82	—	—	—	—	—	3	8	11	8	14	6	4	1	1	—	1	—	—	—	—	—	—	—	—	—	57
83	—	—	—	—	1	1	7	5	10	14	10	11	4	3	—	—	—	—	—	—	—	—	—	—	—	66
84	—	—	—	—	—	—	2	6	9	9	18	15	11	6	8	2	1	—	—	—	—	—	—	—	—	87
85	—	—	—	—	—	—	—	6	5	9	12	11	16	11	5	3	4	—	—	—	—	—	—	—	—	84
86	—	—	—	—	—	—	—	—	2	9	8	22	19	15	14	11	3	2	1	—	—	—	—	—	—	106
87	—	—	—	—	—	—	1	2	1	2	8	11	15	15	9	4	4	3	1	—	—	1	—	—	—	77
88	—	—	—	—	—	—	—	2	—	1	1	7	8	7	16	10	9	5	2	—	—	—	—	—	—	70
89	—	—	—	—	—	—	—	1	1	—	—	1	10	13	7	10	10	7	4	—	—	—	—	—	—	64
90	—	—	—	—	—	—	—	—	—	—	2	3	4	7	4	11	6	10	3	2	—	—	2	—	—	53
91	—	—	—	—	—	—	—	—	—	—	—	3	2	3	4	2	6	4	4	2	4	—	1	—	—	39
92	—	—	—	—	—	—	—	—	—	—	—	1	—	2	3	5	6	4	6	1	1	1	—	—	—	30
93	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	—	3	3	—	2	—	—	1	1	—	13
94	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	1	—	1	1	1	4	—	—	—	—	10
95	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1	1	1	2	1	—	—	—	7
96	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	2	—	—	4
97	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	2
98	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
99	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2
	1	1	2	8	15	9	31	46	56	68	73	89	91	86	73	60	52	40	27	12	12	3	7	3	1	866

TABLE XIII.

Temporal Chord Measurement T_2.

Temporal Bone, T_2. (Central Values.) Right.

Temporal Bone, T_2. (Central Values.) Left.

	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	
35	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
36	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
37	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
38	1	—	—	2	2	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
39	—	—	2	—	—	2	2	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	9
40	—	—	—	1	1	2	4	8	2	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	21
41	—	1	1	—	3	4	6	5	5	—	1	—	—	1	—	1	—	—	—	—	—	—	—	—	27
42	—	—	1	1	3	3	8	9	6	6	7	2	—	—	1	—	1	—	—	—	—	—	—	—	48
43	—	—	—	—	—	1	5	13	9	6	9	2	2	1	—	1	—	—	—	—	—	—	—	—	46
44	—	—	—	—	1	1	4	5	8	19	9	10	5	1	1	1	1	—	—	—	—	—	—	—	66
45	—	—	—	—	—	—	3	2	16	15	8	16	9	4	4	2	1	1	—	—	—	—	—	—	81
46	—	—	—	—	—	—	2	2	8	11	15	16	15	11	9	3	2	—	—	—	—	—	—	—	94
47	—	—	—	—	—	—	—	1	2	11	13	13	16	9	14	3	3	1	—	—	—	—	—	—	91
48	—	—	—	—	—	—	—	1	1	6	8	17	9	14	14	8	5	—	—	—	—	—	—	—	83
49	—	—	—	—	—	—	—	—	—	1	5	7	11	14	8	10	8	4	1	2	1	—	—	—	71
50	—	—	—	—	—	—	—	—	—	2	3	6	4	13	6	11	9	2	3	3	1	1	—	—	76
51	—	—	—	—	—	—	—	—	—	—	1	1	2	6	6	11	12	8	4	5	1	1	—	—	55
52	—	—	—	—	—	—	—	—	—	—	—	2	2	2	4	5	4	6	4	2	1	—	—	—	32
53	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	3	3	2	2	—	3	—	—	16
54	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	2	2	3	1	—	—	—	11
55	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	—	2	2	—	—	—	—	—	—	7
56	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	4	—	—	—	7
57	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	2
58	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	2
59	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	2	1	6	5	10	14	34	46	59	80	77	92	76	76	80	66	51	30	18	16	9	7	—	1	856

TABLE XIV.

Temporal Chord Measurement T_3.

Temporal Bone, T_3. (Central Values.) Right.

Temporal Bone, T_3. (Central Values.) Left.

	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	
54	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
55	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
56	1	2	—	—	1	—	3	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
57	—	—	1	4	—	1	3	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	11
58	—	—	—	1	4	5	2	2	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	17
59	—	—	—	3	4	4	3	3	2	5	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	25
60	—	—	—	2	—	7	7	6	4	8	4	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	41
61	—	—	—	—	2	3	6	5	8	14	7	3	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	51
62	—	—	—	—	1	1	2	4	8	9	8	6	2	3	—	—	1	1	—	—	—	—	—	—	—	—	—	—	46
63	—	—	—	—	1	1	3	4	10	12	7	12	8	3	1	—	1	—	—	—	—	—	—	—	—	—	—	—	63
64	—	—	—	—	—	—	2	2	3	5	14	14	11	6	5	2	—	—	—	—	—	—	—	—	—	—	—	—	64
65	—	—	—	—	—	—	2	1	7	4	12	18	12	9	4	4	4	—	1	—	—	—	—	—	—	—	—	—	78
66	—	—	—	—	—	—	—	—	1	3	6	11	18	23	14	4	4	—	—	—	—	—	—	—	—	—	—	—	84
67	—	—	—	—	—	1	—	3	2	8	9	15	20	9	15	6	3	5	—	—	—	—	—	—	—	—	—	—	96
68	—	—	—	—	—	—	—	—	2	1	3	9	4	12	13	7	5	5	5	2	1	—	—	—	—	—	—	—	64
69	—	—	—	—	—	—	—	—	1	1	6	6	8	7	10	10	9	6	2	1	1	—	—	—	—	—	—	—	68
70	—	—	—	—	—	—	—	—	—	—	2	3	4	7	1	13	5	5	1	—	3	—	—	—	—	—	—	—	44
71	—	—	—	—	—	—	—	—	—	—	—	—	1	3	8	3	8	7	4	2	2	2	1	1	1	—	—	—	41
72	—	—	—	—	—	—	—	—	—	—	—	1	1	2	2	5	4	3	3	2	—	2	1	1	—	—	—	—	23
73	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	—	2	2	3	2	1	1	—	—	1	—	—	—	16
74	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	1	2	2	1	1	—	—	—	—	—	9
75	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	1	1	1	—	1	2	—	1	—	—	—	—	10
76	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	2	—	—	—	—	—	3
77	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	1	—	—	—	2
78	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	—	—	2
79	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
80	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1
81	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	2
	1	3	1	10	14	22	34	28	49	65	69	86	88	84	64	63	55	39	37	17	13	11	7	5	3	2	—	1	871

TABLE XV.

Temporal Arc Measurement T_4.

Temporal Bone, T_4. (Central Values.) Right.

Temporal Bone, T_4. (Central Values.) Left.

	85·75	86·75	87·75	88·75	89·75	90·75	91·75	92·75	93·75	94·75	95·75	96·75	97·75	98·75	99·75	100·75	101·75	102·75	103·75	104·75	105·75	106·75	107·75	108·75	109·75	110·75	111·75	112·75	113·75	114·75	
81·75	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
82·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
83·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
84·75	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
85·75	1	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
86·75	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
87·75	—	—	—	—	1	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
88·75	—	—	—	—	—	—	1	1	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
89·75	—	—	—	1	1	1	4	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8
90·75	—	—	—	—	—	1	2	—	1	2	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
91·75	—	—	—	—	—	4	3	2	—	2	1	1	—	1	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	15
92·75	—	—	—	—	—	—	1	4	—	2	1	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	12
93·75	—	—	—	—	—	1	2	1	3	7	7	—	2	—	2	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	27
94·75	—	—	—	—	—	1	—	2	5	5	6	6	4	2	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	30
95·75	—	—	—	—	—	1	2	—	2	7	10	8	14	3	1	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	50
96·75	—	—	—	—	—	2	—	—	4	6	10	9	12	7	5	3	4	—	1	1	—	—	—	—	—	—	—	—	—	—	64
97·75	—	—	—	—	—	—	1	1	1	3	7	8	17	12	8	7	3	3	1	—	—	—	—	—	—	—	—	—	—	—	72
98·75	—	—	—	—	—	—	—	—	1	—	3	7	7	6	7	10	10	7	3	1	2	—	2	1	—	—	—	—	—	—	60
99·75	—	—	—	—	—	—	—	—	—	—	1	2	7	10	13	9	6	7	3	3	—	1	—	—	—	—	—	—	—	—	59
100·75	—	—	—	—	1	—	—	—	—	—	—	3	6	7	8	6	13	7	3	—	2	2	—	—	—	—	—	—	—	—	58
101·75	—	—	—	—	—	—	—	—	1	—	—	3	7	4	8	9	9	6	7	1	1	—	—	—	—	—	—	—	—	—	54
102·75	—	—	—	—	—	—	—	—	—	2	—	1	—	3	2	9	9	5	10	3	5	2	—	1	—	—	—	—	—	—	47
103·75	—	—	—	—	—	—	—	—	—	—	—	—	2	1	2	4	2	10	3	6	7	2	6	2	—	—	—	—	—	—	45
104·75	—	—	—	—	—	—	—	—	—	—	1	—	—	2	1	2	7	5	5	8	2	6	4	—	—	—	—	—	—	—	43
105·75	—	—	—	—	—	—	—	—	—	—	1	—	—	1	—	—	2	3	1	4	4	2	3	1	2	—	—	—	—	—	24
106·75	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	1	2	—	2	3	4	1	—	—	—	—	—	—	15
107·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	2	2	3	1	—	—	—	—	—	—	12
108·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	—	—	1	—	—	—	—	—	4
109·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	2	—	—	—	1	—	—	—	—	3
110·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	1	—	—	—	—	—	3
111·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
112·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
113·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1
114·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
115·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	1	1	—	1	4	12	18	12	16	34	49	54	78	62	60	61	61	56	34	31	27	25	19	3	3	—	1	2	—	1	726

TABLE XVI.

Temporal Chord Measurement T_5.

Temporal Bone, T_5. (Central Values.) Right.

Temporal Bone, T_5. (Central Values.) Left.

	38·25	38·75	39·25	39·75	40·25	40·75	41·25	41·75	42·25	42·75	43·25	43·75	44·25	44·75	45·25	45·75	46·25	46·75	47·25	47·75	48·25	48·75	49·25	49·75	50·25	50·75	51·25	51·75	52·25	52·75	53·25	53·75	54·25	54·75	
37·75	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
38·25	—	—	—	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
38·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
39·25	—	1	—	1	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
39·75	—	—	—	1	—	—	1	—	—	1	1	—	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
40·25	—	—	—	—	4	3	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
40·75	1	1	—	1	2	—	—	1	1	1	—	2	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	12
41·25	—	—	—	—	—	4	2	2	—	—	—	—	1	1	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	13
41·75	—	—	—	2	1	1	3	2	4	2	1	2	2	2	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	23
42·25	—	—	—	—	—	—	3	4	—	3	3	4	1	3	1	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	25
42·75	—	—	—	—	—	—	1	4	1	2	6	1	2	6	2	2	1	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	30
43·25	—	—	—	1	1	1	—	2	3	1	3	7	1	6	5	3	2	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	38
43·75	—	1	—	—	3	2	—	2	3	3	3	2	7	8	9	5	5	4	—	2	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	51
44·25	—	—	—	—	1	—	3	2	1	4	2	2	7	6	2	3	4	3	3	1	1	1	3	1	—	—	—	—	—	—	—	—	—	—	—	50
44·75	—	—	—	—	—	—	2	—	3	3	6	5	5	5	6	5	5	7	6	3	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	60
45·25	—	—	—	—	—	—	1	—	1	2	5	7	6	4	9	5	4	3	7	4	3	—	1	—	—	—	—	—	—	—	—	—	—	—	—	57
45·75	—	—	—	—	—	—	—	—	1	1	5	7	3	9	6	6	3	6	7	1	4	—	2	—	—	—	—	—	—	—	—	—	—	—	—	62
46·25	—	—	—	—	—	—	1	—	—	3	5	2	3	8	8	6	11	7	3	9	1	2	—	3	—	—	—	—	—	—	—	—	1	—	—	68
46·75	—	—	—	—	—	—	1	1	1	2	6	5	4	4	2	4	11	6	4	3	3	2	4	2	—	1	1	—	—	—	—	—	—	—	—	66
47·25	—	—	—	—	—	—	—	—	—	1	3	—	2	2	2	6	11	6	5	3	5	2	1	1	—	—	—	—	—	—	—	—	—	—	—	50
47·75	—	—	—	—	1	—	—	—	1	1	—	—	2	4	2	1	5	9	1	5	5	2	1	—	—	—	—	—	—	—	—	—	—	—	—	46
48·25	—	—	—	—	—	—	—	—	—	—	1	—	1	4	7	1	5	4	3	1	2	1	4	1	—	—	—	—	—	—	—	—	—	—	1	43
48·75	—	—	—	—	—	—	—	—	—	1	2	—	1	1	3	1	3	2	4	3	2	1	—	1	1	1	—	1	—	—	—	—	—	—	—	—	40
49·25	—	—	—	—	—	—	—	1	—	—	—	—	1	1	2	1	2	5	2	4	3	1	2	1	—	—	—	—	—	—	—	—	—	—	—	—	36
49·75	—	—	—	—	—	—	—	—	—	—	1	—	1	1	—	2	1	2	4	1	2	4	3	1	—	—	—	—	—	—	—	—	—	1	—	—	27
50·25	—	—	—	—	—	—	—	—	1	—	—	1	—	1	—	—	1	1	1	1	—	1	2	2	4	—	—	—	—	—	—	—	—	—	—	—	16
50·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	2	1	1	—	1	2	2	—	1	—	—	—	—	—	—	—	—	—	—	10
51·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	1	—	1	1	1	2	—	—	—	1	—	—	—	—	—	—	—	—	—	9
51·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	1	—	1	3	—	—	—	—	—	—	—	—	—	—	—	—	10
52·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	1	—	—	1	—	—	—	—	—	—	—	—	—	4
52·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	—	—	—	—	1	1	1	—	—	—	—	—	—	—	6
53·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
53·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	1	—	—	—	—	1	—	—	—	—	3
54·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
54·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	2
	1	3	—	6	7	12	19	21	21	30	38	51	51	70	64	60	57	60	56	55	35	32	31	30	18	20	10	7	3	3	1	—	1	2	1	876

TABLE XVII.

Temporal Chord Measurement T_6.

Temporal Bone, T_6. (Central Values.) Right.

Temporal Bone, T_6. (Central Values.) Left.

	33·5	34·5	35·5	36·5	37·5	38·5	39·5	40·5	41·5	42·5	43·5	44·5	45·5	46·5	47·5	48·5	49·5	50·5	51·5	52·5	53·5	54·5	55·5	56·5	57·5	58·5	
33·5	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
34·5	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
35·5	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
36·5	—	1	—	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
37·5	—	—	—	2	1	2	—	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8
38·5	—	—	—	—	—	2	3	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8
39·5	—	—	—	1	—	4	5	4	3	3	—	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	23
40·5	—	—	—	1	1	1	5	5	9	12	4	1	1	—	—	1	—	—	—	—	—	—	—	—	—	—	30
41·5	—	—	—	—	2	3	7	8	8	5	4	5	1	—	1	—	—	—	—	—	—	—	—	—	—	—	44
42·5	—	—	—	—	2	2	6	1	10	12	8	4	5	3	2	—	1	—	—	—	—	—	—	—	—	—	56
43·5	—	—	—	—	1	1	4	3	10	8	16	17	10	9	9	—	—	—	—	—	—	—	—	—	—	—	88
44·5	—	—	—	—	—	—	—	4	9	8	8	16	9	7	3	3	4	—	—	—	—	—	—	—	—	—	73
45·5	—	—	—	—	—	1	—	2	2	3	8	13	16	9	12	6	3	12	—	—	—	—	—	—	—	—	80
46·5		—	—	—	1	—	1	1	4	4	16	9	19	15	10	9	6	3	12	—	—	—	—	—	—	—	100
47·5	—	—	—	—	—	—	—	—	1	4	8	6	11	14	10	11	9	2	12	—	—	—	—	—	—	—	78
48·5	—	—	—	—	—	—	—	—	—	—	3	6	4	8	15	11	6	1	12	12	—	—	—	—	—	—	56
49·5	—	—	—	—	—	—	—	—	—	1	3	3	4	4	—	9	13	8	5	2	1	1	—	1	—	—	63
50·5	—	—	—	—	—	—	—	—	—	—	1	2	2	5	6	4	13	3	7	1	—	—	—	—	—	—	44
51·5	—	—	—	—	—	—	—	—	—	—	—	—	2	—	5	5	2	4	3	4	2	—	—	—	—	—	27
52·5	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	1	1	2	1	3	—	1	1	—	—	—	12
53·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	3	1	3	1	2	—	1	—	—	—	13
54·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	2	1	—	—	—	—	—	4
55·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	2	—	—	—	—	—	4
56·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	1	1	—	2	6
57·5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1
	1	2	—	4	9	17	32	32	56	53	80	86	84	74	84	61	62	29	25	17	9	3	3	2	—	2	827

Sphenoid Bone, S_2. (Central Values.) Left.

TABLE XVIII.

Temporal Chord Measurement T_7.

Temporal Bone, T_7. (Central Values.) Right.

Temporal Bone, T_7. (Central Values.) Left.

	27·25	27·75	28·25	28·75	29·25	29·75	30·25	30·75	31·25	31·75	32·25	32·75	33·25	33·75	34·25	34·75	35·25	35·75	36·25	36·75	37·25	37·75	38·25	38·75	39·25	39·75	40·25	40·75	41·25	41·75	42·25	42·75	43·25	43·75	44·25	44·75	
26·25	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
26·75	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
27·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
27·75	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
28·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	1
28·75	1	—	—	—	—	1	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
29·25	—	—	—	—	—	1	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
29·75	—	—	—	—	1	—	—	1	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
30·25	—	—	—	1	2	—	—	1	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
30·75	—	—	—	1	1	—	1	1	5	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	11
31·25	—	1	1	—	1	1	2	1	—	3	1	—	2	1	1	—	1	—	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	18
31·75	—	—	—	—	4	1	1	—	3	2	—	2	3	1	2	1	1	1	—	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	22
32·25	—	—	—	—	—	1	—	—	3	1	1	3	2	2	2	—	1	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	36
32·75	1	—	—	—	—	—	1	—	1	2	3	4	5	4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	35
33·25	—	—	—	—	1	—	—	—	3	3	2	6	3	2	6	3	4	2	1	—	2	2	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	33
33·75	—	—	—	—	—	—	1	3	3	2	4	2	3	6	1	2	4	—	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	41
34·25	—	—	—	—	—	—	—	—	3	4	3	7	7	5	5	4	3	1	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	48
34·75	—	—	—	—	—	—	1	—	1	2	3	4	7	5	5	5	6	3	4	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	56
35·25	—	—	—	—	—	1	—	—	2	—	1	1	5	3	9	7	8	7	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	52
35·75	—	—	—	—	—	—	1	—	—	1	1	3	4	7	12	4	6	4	3	3	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	56
36·25	—	—	—	—	—	—	1	1	—	—	3	2	3	4	5	4	8	11	7	4	3	3	—	—	1	—	—	—	—	—	—	—	—	—	—	—	63
36·75	—	—	—	—	—	—	—	1	—	—	—	—	3	6	4	9	7	10	7	11	7	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	72
37·25	—	—	—	—	—	—	—	—	—	—	—	2	—	1	4	4	8	12	4	7	6	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	50
37·75	—	—	—	—	—	—	—	—	—	—	1	1	1	—	2	4	9	6	5	3	7	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	47
38·25	—	—	—	—	—	—	—	—	—	—	2	—	—	1	—	3	4	6	5	6	2	6	—	—	—	—	—	—	—	—	—	—	—	—	—	—	40
38·75	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	2	1	3	1	7	2	6	—	—	—	—	—	—	—	—	—	—	—	—	—	—	36
39·25	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	2	1	—	1	1	1	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	27
39·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	29
40·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	17
40·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	3	1	—	—	—	—	—	—	—	—	—	—	—	—	17
41·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	10
41·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
42·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
42·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	3
43·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
43·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
44·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
44·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
	2	—	2	3	4	8	6	9	8	29	23	25	32	46	42	54	59	56	57	60	42	51	40	41	26	36	21	16	11	8	10	6	5	2			840

Sphenoid Bone, S_3. (Central Values.) Left.

TABLE XIX.

Sphenoidal Chord Measurement S_2.

Sphenoid Bone, S_2. (Central Values.) Right.

Sphenoid Bone, S_2. (Central Values.) Left.

	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	
65	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
66	—	—	1	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
67	—	—	—	2	2	2	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8
68	—	2	—	1	3	4	3	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	17
69	—	—	—	—	1	5	3	3	3	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	17
70	—	—	—	1	—	4	6	6	7	4	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	31
71	—	—	—	1	1	2	6	3	8	12	2	5	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	42
72	—	—	—	—	—	—	5	7	20	8	14	3	5	1	2	1	—	—	—	—	—	—	—	—	—	—	—	66
73	—	—	—	—	—	—	4	8	14	21	18	12	9	2	—	—	—	—	—	—	—	—	—	—	—	—	—	88
74	—	—	—	—	—	—	2	2	3	13	20	11	10	8	4	1	—	—	—	—	—	—	—	—	—	—	—	74
75	—	—	—	—	—	—	1	1	5	12	10	18	9	7	4	4	2	—	—	—	—	—	—	—	—	—	—	73
76	—	—	—	—	—	—	—	—	1	5	9	6	24	12	8	4	2	1	1	—	—	—	—	—	—	—	—	73
77	—	—	—	—	—	—	—	—	—	1	8	11	10	19	5	7	3	4	2	—	—	—	—	—	—	—	—	70
78	—	—	—	—	—	—	—	—	—	1	2	6	7	6	15	5	3	3	—	—	—	—	—	—	—	—	—	48
79	—	—	—	—	—	—	—	—	—	—	—	3	3	6	8	19	4	1	1	—	—	—	—	—	—	—	—	45
80	—	—	—	—	—	—	—	—	—	—	1	—	1	3	5	1	6	—	1	2	1	—	—	—	—	—	—	21
81	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2	3	4	6	3	1	—	—	—	—	—	—	—	21
82	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	2	5	—	2	—	—	—	—	—	—	12
83	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	—	—	—	1	2	—	—	—	—	—	6
84	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	—	—	—	—	—	—	—	2
85	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
86	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
87	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
88	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
89	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
90	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	1	2	1	5	8	17	30	35	62	80	85	77	80	66	54	47	28	17	13	4	4	2	—	—	—	—	1	719

TABLE XX.

Sphenoidal Chord Measurement S_3.

Sphenoid Bone, S_3. (Central Values.) Right.

Sphenoid Bone, S_3. (Central Values.) Left.

	30·25	30·35	31·45	32·05	32·65	33·25	33·85	34·45	35·05	35·65	36·25	36·85	37·45	38·05	38·65	39·25	39·85	40·45	41·05	41·65	42·25	42·85	43·45	
30·25	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
30·35	—	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
31·45	1	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
32·05	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
32·65	—	1	2	2	—	4	4	1	3	1	1	—	—	—	—	—	—	—	—	—	—	—	—	20
33·25	1	—	—	2	4	3	7	4	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	23
33·85	—	1	—	3	3	3	6	14	4	4	3	1	—	1	—	—	—	—	—	—	—	—	—	43
34·45	—	—	1	2	5	5	8	13	7	12	11	—	2	—	—	—	—	—	—	—	—	—	—	64
35·05	—	—	—	1	1	2	7	11	24	5	16	3	1	—	1	—	—	—	—	—	—	—	—	72
35·65	—	—	—	1	—	3	6	11	18	17	19	16	3	4	—	1	—	—	—	—	—	—	—	99
36·25	—	—	—	1	2	2	4	10	9	16	32	15	10	3	1	2	—	—	—	—	—	—	—	107
36·85	—	—	—	—	1	1	2	1	4	16	21	19	13	11	2	2	—	—	—	—	—	—	—	93
37·45	—	—	—	—	—	—	2	—	2	10	20	22	23	15	3	2	2	1	—	—	—	—	—	102
38·05	—	—	—	—	—	—	—	—	2	6	5	14	13	18	8	6	1	—	—	—	—	—	—	73
38·65	—	—	—	—	—	—	—	—	—	3	3	8	8	6	5	4	1	—	1	—	—	—	—	39
39·25	—	—	—	—	—	—	—	2	4	3	3	8	3	12	5	8	3	3	—	—	—	—	—	51
39·85	—	—	—	—	—	—	—	—	—	—	—	2	6	4	7	10	—	2	3	1	—	—	—	35
40·45	—	—	—	—	—	—	—	—	—	—	—	1	2	1	3	3	1	7	1	—	—	—	—	19
41·05	—	—	—	—	—	—	—	—	—	—	—	1	1	1	2	4	—	—	1	—	—	—	1	11
41·65	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1	—	—	—	2
42·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1
42·85	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1
43·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1
	2	4	5	15	15	25	46	65	76	94	135	110	85	76	38	42	8	13	8	3	—	—	1	866

TABLE XXI.

Sphenoidal Chord Measurement S_5.

Sphenoid Bone, S_5. (Central Values.) Right.

Sphenoid Bone, S_5. (Central Values.) Left.	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	
46	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
47	—	1	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
48	—	—	2	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
49	—	—	1	1	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
50	—	—	2	3	6	1	2	—	—	—	—	2	1	—	—	—	—	—	—	—	—	—	17
51	1	—	2	1	6	2	2	4	—	—	2	1	—	—	—	—	—	—	—	—	—	—	21
52	—	—	1	2	6	9	14	8	6	2	1	—	1	—	—	—	—	—	—	—	—	—	50
53	—	—	—	4	3	7	9	8	7	6	4	2	1	1	—	—	—	—	—	—	—	—	52
54	—	—	—	—	3	5	14	16	13	13	6	2	2	2	—	—	—	—	—	—	—	—	76
55	—	—	—	—	1	3	4	14	13	16	9	6	2	4	—	2	—	—	—	—	—	—	74
56	—	—	—	2	—	3	4	7	12	16	15	11	10	2	1	—	—	—	—	—	—	—	83
57	—	—	—	—	—	1	2	3	5	14	11	18	12	6	4	—	—	—	—	—	—	—	76
58	—	—	—	—	—	—	—	3	4	9	13	10	10	13	2	3	—	—	—	—	—	—	67
59	—	—	—	—	—	—	1	3	1	3	8	10	14	9	9	—	1	—	—	—	—	—	59
60	—	—	—	—	—	—	—	1	—	3	4	5	10	10	14	7	—	2	—	—	—	—	56
61	—	—	—	—	—	—	—	1	—	1	1	4	6	3	10	1	6	—	—	—	—	—	33
62	—	—	—	—	—	—	—	—	—	1	—	3	1	6	3	2	3	1	2	—	—	—	22
63	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	3	1	—	—	—	—	7
64	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	2	3	1	—	—	—	8
65	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	1	1	1	—	—	5
66	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	—	2
67	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
68	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1	2
	2	1	8	14	26	34	52	69	61	84	74	74	70	57	46	17	16	10	4	1	1	1	722

TABLE XXII.

Sphenoidal Chord Measurement S_6.

Sphenoid Bone, S_6. (Central Values.) Right.

Sphenoid Bone, S_6. (Central Values.) Left.	28·45	29·05	29·65	30·25	30·85	31·45	32·05	32·65	33·25	33·85	34·45	35·05	35·65	36·25	36·85	37·45	38·05	38·65	39·25	39·85	40·45	41·05	41·65	42·25	42·85	
28·45	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
29·05	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
29·65	—	—	1	—	—	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
30·25	—	—	1	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
30·85	1	1	1	—	3	4	2	1	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	16
31·45	—	—	—	4	—	5	6	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	18
32·05	—	—	—	—	2	1	8	5	4	6	3	1	1	—	—	—	—	—	—	—	—	—	—	—	—	31
32·65	—	—	—	—	2	4	4	8	6	6	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	32
33·25	—	—	—	—	—	—	2	11	23	11	5	4	4	1	1	—	—	—	—	—	—	—	—	—	—	62
33·85	—	—	—	—	—	1	4	9	10	17	10	10	4	2	1	—	—	—	—	—	—	—	—	—	—	68
34·45	—	—	—	—	—	—	1	—	6	11	15	10	15	4	1	1	1	—	—	—	—	—	—	—	—	65
35·05	—	—	—	—	—	—	1	4	7	7	16	18	15	11	6	2	4	1	2	—	—	—	—	—	—	94
35·65	—	—	—	—	—	—	—	—	—	5	8	13	21	16	6	3	1	—	—	—	—	—	—	—	—	73
36·25	—	—	—	—	—	—	—	—	—	3	8	13	13	22	9	3	2	1	—	—	—	—	—	—	—	75
36·85	—	—	—	—	—	—	—	—	—	—	2	9	10	13	13	8	7	6	2	1	—	—	—	—	—	71
37·45	—	—	—	—	—	—	—	—	—	—	—	—	2	8	10	7	10	7	3	2	—	—	—	—	—	49
38·05	—	—	—	—	—	—	—	—	—	—	—	—	4	3	5	6	14	2	5	1	1	—	—	—	—	37
38·65	—	—	—	—	—	—	—	—	—	—	—	—	—	3	1	4	8	13	2	—	1	1	—	—	—	33
39·25	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	4	2	3	7	5	4	1	—	—	—	27
39·85	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	1	3	5	1	4	—	—	1	17
40·45	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2	1	3	1	2	1	—	—	11
41·05	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	4	1	—	—	—	—	6
41·65	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	1	—	1	—	1	—	5
42·25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	1	3
42·85	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	2
	1	2	4	7	9	18	29	40	57	69	68	79	89	85	54	41	52	36	23	21	9	9	2	4		808

TABLE XXIII.

Malar Arc Measurement Ml_1.

Malar Bone, Ml_1. (Central Values.) Right.

Malar Bone, Ml_2. (Central Values.) Left.	37·75	38·75	39·75	40·75	41·75	42·75	43·75	44·75	45·75	46·75	47·75	48·75	49·75	50·75	51·75	52·75	53·75	54·75	55·75	56·75	57·75	58·75	59·75	60·75	61·75	
37·75	1																									1
38·75																										—
39·75				1																						1
40·75																										—
41·75				1	2	1																				4
42·75				2	3	4	3	2																		14
43·75						5	3	3	1																	12
44·75							4	5	6	1	2															18
45·75						1	4	13	15	9	3	1														46
46·75							2	8	23	18	13	2	1													67
47·75								3	14	17	29	15	4	2												84
48·75									5	10	20	29	15	2	1											82
49·75									1	6	16	29	34	14	1											101
50·75										2	1	8	40	41	13	3										108
51·75											1	5	17	27	23	15	2									90
52·75											1	3	4	16	17	22	6	2	1							72
53·75												1	5	5	8	11	13	1								44
54·75														2	6	6	9	7	3							33
55·75														1		3	2	8	7	1						22
56·75																	1	2	2	2	1					9
57·75																			2	1	1					4
58·75																				1	1					2
59·75																										—
60·75																					1	1				2
61·75																						1				1
	1	—	—	4	5	11	16	34	65	63	86	93	120	110	69	60	33	20	15	5	2	4	1			817

TABLE XXIV.

Malar Arc Measurement Ml_2.

Malar Bone, Ml_2. (Central Values.) Right.

Malar Bone, Ml_2. (Central Values.) Left.	37·75	38·75	39·75	40·75	41·75	42·75	43·75	44·75	45·75	46·75	47·75	48·75	49·75	50·75	51·75	52·75	53·75	54·75	55·75	56·75	57·75	58·75	59·75	60·75	61·75	62·75	63·75	64·75	65·75	66·75	67·75	68·75	69·75	70·75	71·75	72·75	73·75	
37·75	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
38·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
39·75	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
40·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
41·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
42·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
43·75	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
44·75	—	—	—	—	—	—	—	—	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
45·75	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
46·75	—	—	—	—	—	—	—	—	—	1	—	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
47·75	—	—	—	—	—	—	—	—	—	—	1	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
48·75	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
49·75	—	—	—	—	—	—	—	—	—	—	—	1	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
50·75	—	—	—	—	—	—	—	—	—	—	—	—	1	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
51·75	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	14
52·75	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	17
53·75	—	—	—	—	—	—	—	—	—	—	—	—	1	—	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	24
54·75	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	29
55·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	43
56·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	50
57·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	81
58·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	62
59·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	58
60·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	57
61·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	58
62·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	54
63·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	44
64·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	38
65·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	28
66·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	14
67·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	12
68·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	10
69·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
70·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
71·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
72·75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
	1	—	—	1	—	—	1	—	—	2	3	3	4	8	7	23	24	38	38	53	71	70	61	69	61	55	37	28	18	21	7	5	2	4	2	—	1	718

TABLE XXV.

Maxillary Chord Measurement Mx_1.

Maxillary Bone, Mx_1. (Central Values.) Right.

Maxillary Bone, Mx_1. (Central Values.) Left.

	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	
52	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
53	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
54	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
55	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
56	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
57	—	—	—	—	1	1	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5
58	—	—	—	—	—	5	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
59	—	—	—	—	1	2	3	4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	10
60	—	—	—	—	—	—	5	2	4	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	21
61	—	—	—	—	—	—	—	2	12	7	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	21
62	—	—	—	—	—	—	—	—	3	19	5	3	2	—	—	—	—	—	—	—	—	—	—	—	—	—	32
63	—	—	—	—	—	—	—	—	1	9	26	10	1	1	—	—	—	—	—	—	—	—	—	—	—	—	48
64	—	—	—	—	—	—	—	—	—	1	12	28	19	3	2	—	—	—	—	—	—	—	—	—	—	—	65
65	—	—	—	—	—	—	—	—	—	—	1	10	22	24	5	1	—	—	—	—	—	—	—	—	—	—	63
66	—	—	—	—	—	—	—	—	—	—	—	2	9	29	25	3	—	—	—	—	—	—	—	—	—	—	68
67	—	—	—	—	—	—	—	—	—	—	—	—	2	6	36	21	6	2	—	—	—	—	—	—	—	—	73
68	—	—	—	—	—	—	—	—	—	—	—	—	2	1	11	28	17	3	1	—	—	—	—	—	—	—	63
69	—	—	—	—	—	—	—	—	—	—	—	—	—	—	5	12	26	22	—	2	—	—	—	—	—	—	67
70	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8	20	12	2	—	—	—	—	—	—	42
71	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	9	20	7	3	—	—	—	—	—	41
72	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	16	6	2	—	—	—	—	27
73	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	2	5	1	—	—	—	11
74	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	6	1	—	—	—	10
75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	2	—	—	—	4
76	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	—	—	3
77	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	2	1	4
	2	—	3	—	4	8	12	16	20	39	44	53	57	64	84	65	59	56	36	30	14	15	5	3	2	1	692

TABLE XXVI.

Maxillary Chord Measurement Mx_2.

Maxillary Bone, Mx_2. (Central Values.) Right.

Maxillary Bone, Mx_2. (Central Values.) Left.

	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	
47	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
48	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
49	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
50	—	—	—	1	7	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	10
51	—	—	—	1	5	6	5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	17
52	—	—	—	—	3	9	8	10	1	1	—	—	—	—	—	—	—	—	—	—	—	—	32
53	—	—	—	—	1	6	8	7	11	1	5	—	—	—	—	—	—	—	—	—	—	—	39
54	—	—	—	—	—	—	6	7	20	11	13	—	—	—	—	—	—	—	—	—	—	—	57
55	—	—	—	—	—	—	1	4	10	34	15	1	—	—	—	—	—	—	—	—	—	—	65
56	—	—	—	—	—	—	—	1	12	18	37	10	—	—	—	—	—	—	—	—	—	—	78
57	—	—	—	—	—	—	—	—	1	8	13	26	7	2	—	—	—	—	—	—	—	—	57
58	—	—	—	—	—	—	—	—	—	2	9	17	23	3	7	—	—	—	—	—	—	—	61
59	—	—	—	—	—	—	—	—	—	—	1	8	9	15	12	—	—	—	—	—	—	—	45
60	—	—	—	—	—	—	—	—	—	—	—	1	5	11	5	1	—	—	—	—	—	—	23
61	—	—	—	—	—	—	—	—	—	—	—	—	1	1	5	3	3	—	—	—	—	—	15
62	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	3	2	—	1	—	—	—	8
63	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	2
64	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	2
65	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
66	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
67	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	2	—	1	3	17	22	29	29	55	75	93	63	45	33	30	10	7	1	1	—	—	1	517

TABLE XXVII.

Maxillary Chord Measurement Mx_3.

Maxillary Bone, Mx_3. (Central Values.) Right.

Maxillary Bone, Mx_3. (Central Values.) Left.

	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	
31	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	1
32	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
33	1	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
34	—	—	—	—	4	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
35	—	—	—	—	1	3	1	3	1	—	—	—	—	—	—	—	—	—	—	—	9
36	—	—	—	—	3	3	2	8	3	—	—	—	—	—	—	—	—	—	—	—	17
37	—	—	—	—	1	—	5	13	8	2	—	1	—	—	—	—	—	—	—	—	30
38	—	—	—	—	—	1	1	8	13	9	7	—	—	—	—	—	—	—	—	—	39
39	—	—	—	—	—	—	—	4	5	21	8	1	3	1	—	—	—	—	—	—	43
40	—	—	—	—	—	—	1	2	7	6	25	13	5	—	—	—	—	—	—	—	59
41	—	—	—	—	—	—	—	—	2	4	14	15	17	6	2	—	—	—	—	—	60
42	—	—	—	—	—	—	—	—	—	1	8	11	16	7	2	—	—	—	—	—	45
43	—	—	—	—	—	—	—	—	—	1	3	3	5	12	13	2	—	—	—	—	39
44	—	—	—	—	—	—	—	—	—	—	1	—	3	11	14	4	3	—	—	—	36
45	—	—	—	—	—	—	—	—	—	—	—	1	3	4	8	9	5	—	—	—	30
46	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	1	3	4	3	1	15
47	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3	3	2	—	—	8
48	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	1	—	3
49	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	1	3	6
50	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
51	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	1	—	—	1	8	10	11	38	39	44	66	45	52	43	41	21	15	6	5	5	451

Occipital Bone, O_y. (Central Values.) Left.

TABLE XXVIII.

Maxillary Chord Measurement Mx_4.

Maxillary Bone, Mx_4. (Central Values.) Right.

Maxillary Bone, Mx_4. (Central Values.) Left.

	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	
30	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
31	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
32	—	1	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
33	—	—	1	7	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9
34	—	—	6	6	6	4	1	1	—	—	—	—	—	—	—	—	—	—	—	—	24
35	—	2	1	5	5	14	4	2	—	—	—	—	—	—	—	—	—	—	—	—	33
36	—	—	—	4	6	16	12	6	2	3	—	—	—	—	—	—	—	—	—	—	49
37	—	—	—	1	6	17	32	27	7	6	1	2	—	—	—	—	—	—	—	—	99
38	—	—	—	—	2	7	16	29	14	10	—	—	—	—	—	—	—	—	—	—	78
39	—	—	—	—	1	2	6	15	25	15	7	2	—	—	—	—	—	—	—	—	73
40	—	—	—	—	—	—	4	6	15	20	10	4	1	—	—	—	—	—	—	—	60
41	—	—	—	—	—	—	—	—	6	12	18	9	—	—	—	—	—	—	—	—	45
42	—	—	—	—	—	—	—	2	6	6	8	13	3	1	1	—	—	—	—	—	40
43	—	—	—	—	—	—	—	—	1	2	1	1	4	2	—	—	—	—	—	—	11
44	—	—	—	—	—	—	—	—	—	—	2	4	—	1	2	—	—	—	—	—	9
45	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	1	—	—	—	—	4
46	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	2
47	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	2
48	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
49	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1
	2	3	10	23	27	61	75	88	76	74	47	35	10	5	5	1	2	—	—	1	545

TABLE XXIX.

Occipital Arc Measurement O_7.

Occipital Bone, O_7. (Central Values.) Right.

Occipital Bone, O_7. (Central Values.) Left.

	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	
85	1	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
86	1	1	—	—	2	2	—	—	2	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9
87	—	—	2	—	—	—	2	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
88	—	—	—	—	3	—	—	—	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	6
89	1	—	—	—	1	2	1	1	3	1	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	14
90	—	—	—	—	3	2	1	1	4	2	1	—	2	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	18
91	—	—	1	1	3	2	3	2	3	1	4	2	1	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	25
92	—	1	1	—	1	2	3	3	4	4	2	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	23
93	—	—	—	1	4	6	2	3	7	3	6	2	3	2	2	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	43
94	—	—	1	2	—	1	5	4	4	3	5	6	3	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	36
95	—	—	—	—	—	3	3	4	9	7	13	6	6	2	4	6	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	64
96	—	—	—	—	—	2	1	4	3	—	11	4	7	7	6	3	1	1	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	52
97	—	—	—	—	—	—	1	—	1	3	7	4	5	3	6	2	4	1	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	39
98	—	—	—	1	—	1	—	—	3	5	8	11	5	7	10	5	3	2	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	65
99	—	—	—	—	—	1	—	—	2	1	5	5	6	9	8	2	6	1	2	2	3	—	—	—	—	—	—	—	—	—	—	—	—	—	53
100	—	—	—	—	—	—	—	—	1	1	2	5	3	3	7	4	13	4	5	4	2	2	—	1	—	—	—	—	—	—	—	—	—	—	55
101	—	—	—	—	—	—	—	—	1	1	—	4	3	3	6	4	6	7	7	8	4	6	2	1	—	—	—	—	—	—	—	—	—	—	68
102	—	—	—	—	—	—	—	—	—	—	1	1	5	8	3	8	6	7	7	8	4	6	2	1	—	—	—	1	—	—	—	—	—	—	68
103	—	—	—	—	—	—	—	—	—	2	1	1	3	7	4	3	4	6	3	2	2	1	—	1	—	—	—	—	—	—	—	—	—	—	40
104	—	—	—	—	—	—	—	—	—	—	—	1	3	4	2	2	3	2	6	3	1	1	4	—	—	—	—	—	—	—	—	—	—	—	—	34
105	—	—	—	—	—	—	—	—	—	—	—	—	3	—	3	1	3	7	7	4	6	4	3	2	—	—	1	—	—	—	—	—	—	—	44
106	—	—	—	—	—	—	—	—	—	—	—	—	3	—	2	1	3	3	6	2	2	1	—	1	2	—	—	1	—	—	—	—	—	—	24
107	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	2	3	2	3	2	2	2	1	1	—	—	—	—	—	—	—	20
108	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	1	—	—	3	1	2	2	3	—	—	—	2	—	—	—	—	—	—	—	16
109	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	2	—	—	—	3	2	—	2	3	—	—	—	—	—	—	—	—	—	13
110	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	5
111	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	1	—	—	—	—	—	—	—	—	—	3
112	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	1	—	—	—	1	—	—	—	—	1	—	—	4
113	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	2
114	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	—	—	—	—	—	2
115	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	1	1	—	2
116	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
117	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
118	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
119	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
120	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1
	3	2	5	6	18	24	22	26	49	36	68	54	65	61	69	49	54	48	51	39	28	27	21	13	7	6	3	4	1	1	—	—	3	1	864

TABLE XXX.

Occipital Arc Measurement O_8.

Occipital Bone, O_8. (Central Values.) Right.

Occipital Bone, O_8. (Central Values.) Left.	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	
54	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
55	—	—	—	1	—	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	4
56	—	—	—	—	—	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2
57	—	—	—	—	1	1	1	1	1	1	1	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	9
58	—	—	—	—	—	—	1	2	6	6	6	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	23
59	1	—	—	—	—	1	3	4	5	4	7	2	6	3	1	—	—	—	—	—	—	—	—	—	—	—	—	—	37
60	—	—	—	—	1	1	7	7	11	15	11	8	8	6	3	4	2	—	—	—	—	1	—	—	—	—	—	—	85
61	—	—	—	—	2	4	2	7	5	13	7	8	14	5	4	4	—	1	—	2	—	—	—	—	—	—	—	—	78
62	—	—	—	—	—	1	4	8	6	11	17	15	12	11	7	3	2	1	1	—	—	—	2	—	1	—	—	—	102
63	—	—	—	1	—	—	3	1	3	10	21	17	19	12	11	6	3	1	1	—	—	2	1	1	—	—	—	—	113
64	—	—	—	—	—	1	—	4	5	9	11	9	11	7	10	4	1	1	1	2	—	—	—	1	—	—	—	1	78
65	—	—	—	1	—	—	1	1	4	3	5	11	5	14	12	6	7	2	1	2	1	1	—	1	1	—	—	—	79
66	—	—	—	—	—	—	—	1	1	2	9	8	10	14	8	11	4	5	1	—	—	1	—	1	—	—	—	—	76
67	—	—	—	—	—	—	1	—	—	1	1	2	14	7	6	7	11	6	4	4	2	—	—	—	—	—	—	—	66
68	—	—	—	—	—	—	—	1	—	—	1	—	2	5	6	5	5	5	3	1	3	—	—	—	—	—	—	—	37
69	—	1	—	—	—	—	—	—	1	—	2	2	3	3	3	2	2	8	2	2	—	1	—	—	—	—	—	—	32
70	—	—	—	1	—	—	—	—	—	—	—	1	1	2	1	1	5	1	4	1	—	—	—	—	—	—	—	—	18
71	—	—	—	—	—	—	—	—	—	—	—	—	1	1	—	1	1	2	1	1	—	1	—	—	—	—	—	—	9
72	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	1	2	—	—	—	—	—	—	—	5
73	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1	1	—	—	—	—	—	—	—	—	3
74	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
75	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
76	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
77	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1
	1	1	—	3	5	13	23	37	49	75	100	83	106	91	75	55	44	33	21	17	8	7	3	4	2	—	1	1	858

TABLE XXXI.

Occipital Chord Measurement O_9.

Occipital Bone, O_9. (Central Values.) Right.

Occipital Bone, O_9. (Central Values.) Left.	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	
64	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1
65	1	—	1	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
66	—	1	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	3
67	3	1	1	1	—	—	—	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	9
68	—	2	1	2	4	2	2	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	17
69	—	1	1	3	2	1	4	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	15
70	—	—	—	2	6	7	7	6	3	5	2	2	1	—	—	—	—	—	—	—	—	—	41
71	—	—	1	—	1	13	10	11	10	8	6	3	—	1	—	—	—	—	—	—	—	—	64
72	—	—	1	1	2	4	14	24	12	8	14	3	—	—	—	—	—	—	—	—	—	—	83
73	—	—	—	1	3	9	11	21	22	8	14	11	6	—	—	—	—	—	—	—	—	—	106
74	—	—	1	—	2	2	7	7	14	20	20	21	8	4	1	2	—	—	—	—	—	—	109
75	—	—	—	—	—	—	3	5	9	16	16	20	16	11	3	3	—	—	—	—	—	—	102
76	—	—	—	—	—	—	—	1	6	7	17	18	16	4	7	2	3	—	1	—	—	—	82
77	—	—	—	—	—	—	—	3	—	8	10	13	17	15	6	4	—	—	—	—	—	—	76
78	—	—	—	—	—	—	—	1	2	—	—	7	13	5	11	6	8	6	—	—	—	—	59
79	—	—	—	—	—	—	—	—	1	—	1	3	9	3	4	9	8	—	—	—	—	—	38
80	—	—	—	—	—	—	—	—	—	—	—	1	—	2	3	4	4	2	—	—	—	—	16
81	—	—	—	—	—	—	—	—	—	—	—	2	1	1	1	4	4	2	1	—	—	—	16
82	—	—	—	—	—	—	—	—	—	—	1	1	—	—	1	3	1	—	—	1	1	—	9
83	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	—	—	1	1	—	4
84	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	2
85	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
86	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	2
87	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1
	4	5	8	10	22	38	59	87	81	80	108	110	80	52	33	40	26	4	3	2	3	3	858

THE MEAN AND SECOND MOMENT COEFFICIENT OF THE MULTIPLE CORRELATION COEFFICIENT, IN SAMPLES FROM A NORMAL POPULATION.

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Our knowledge regarding the sampling distribution of the multiple correlation coefficient has been very greatly increased in recent years. It has been known since 1924 that, for the special case of zero correlation in the universe, the distribution of R for samples from a normal population is given by*

$$df = \frac{(a+b-1)!}{(a-1)!\,(b-1)!}(R^2)^{a-1}(1-R^2)^{b-1}\,d(R^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where a is put, for convenience, for one-half the number of degrees of freedom due to the regression function (i.e. the number of independent variates), and b for one-half the number of degrees of freedom due to deviations from the regression function (i.e. the total number in the sample less the total number of variates). Tables exist for determining the probability of occurrence of a given R from this distribution, and extend to six independent variates and for a size of sample of about 100†. More recently the general distribution of R has been reached by Dr R. A. Fisher‡, and a table, appropriate for large samples, has been furnished whereby the experimenter may, by suitable transformations, determine approximately the significance of an observed R in relation to a given multiple correlation in the universe, exact account being taken of the positive bias of small observed multiple correlations. It is an interesting mathematical exercise, not altogether devoid of practical interest, to use Fisher's distribution to determine the exact nature of this bias, i.e. the amount by which the mean value of R (or R^2, which as we shall see is the more amenable to analysis) is in excess of the true correlation ρ (or ρ^2) existing in the universe. The purpose of the first section of this paper is to determine the mean value of R^2. Later, the analysis is extended to the derivation of the second moment coefficient, or variance, of R^2, although the utility of this quantity, for a distribution which is far from normal, is not so great as would at first sight appear. In both cases the results are compared with Hall's large sample approximations§.

* R. A. Fisher, *Phil. Trans.* B, Vol. 213, 1924, pp. 89—142.

† J. Wishart, *Quart. Journ. Roy. Met. Soc.* Vol. LIV. 1928, pp. 258—259.

‡ R. A. Fisher, *Proc. Roy. Soc.* A, Vol. 121, 1928, pp. 654—673.

§ P. Hall, *Biometrika*, Vol. XIX. 1927, pp. 100—109.

Fisher's general distribution is

$$df = \frac{(a+b-1)!}{(a-1)!\,(b-1)!}(1-\rho^2)^{a+b}\,.\,F(a+b,\,a+b,\,a,\,\rho^2R^2)\,.\,(R^2)^{a-1}(1-R^2)^{b-1}\,d(R^2) \quad \text{.....(2),}$$

and he notes that, when $2b$ is even, we may use the Euler transformation of the hypergeometric function to obtain the distribution in the form

$$df = \frac{(a+b-1)!}{(a-1)!\,(b-1)!}\frac{(1-\rho^2)^{a+b}}{(1-\rho^2R^2)^{a+2b}}F(-b,\,-b,\,a,\,\rho^2R^2)\,.\,(R^2)^{a-1}(1-R^2)^{b-1}\,d(R^2) \quad \text{.....(3),}$$

giving a terminating series.

The fact that, for $2b$ even, he has given the probability integral enables us without a great deal of difficulty to determine for the special cases $b=1$, 2 and 3 the first and second moments of the distribution (3), and thence to infer the general result for any b, which is probably true without any restrictions as to whether $2b$ is even or odd.

A. *Determination of Mean Value of* R^2.

We may conveniently put $R^2\rho^2 = x$.

Case 1. $b=1$. The distribution is

$$df = \frac{(1-\rho^2)^{a+1}}{(\rho^2)^a}\frac{(a+x)\,x^{a-1}}{(1-x)^{a+2}}\,dx \quad \text{.....(4).}$$

We now multiply by R^2, i.e. by x/ρ^2, and integrate with respect to x from 0 to ρ^2. Noting that the indefinite integral of (4) is

$$\frac{(1-\rho^2)^{a+1}}{(\rho^2)^a}\frac{x^a}{(1-x)^{a+1}},$$

we have

$$\bar{R}^2 = \left(\frac{1-\rho^2}{\rho^2}\right)^{a+1}\int_0^{\rho^2} x\,d\left(\frac{x^a}{(1-x)^{a+1}}\right) = 1-\left(\frac{1-\rho^2}{\rho^2}\right)^{a+1}\int_0^{\rho^2}\frac{x^a\,dx}{(1-x)^{a+1}} \quad \text{...(5),}$$

on integrating by parts. Leaving the result in this form meantime we shall consider other cases.

Case 2. $b=2$. The distribution is

$$df = \frac{(1-\rho^2)^{a+2}}{(\rho^2)^a}\cdot\frac{\{a(a+1)+4(a+1)x+2x^2\}\,x^{a-1}(1-x/\rho^2)}{(1-x)^{a+4}}\,dx \text{.....(6),}$$

and the indefinite integral

$$\frac{(1-\rho^2)^{a+2}}{(\rho^2)^a}\left\{\frac{(a+2)(1-x/\rho^2)}{(1-x)^{a+3}} - \frac{1-2x}{(1-x)}\right\}x^a,$$

which may be written

$$\frac{(1-\rho^2)^{a+2}}{(\rho^2)^a}\left\{\frac{(a+1+x)(1-x/\rho^2)}{(1-x)^{a+3}} + \frac{x/\rho^2}{(1-x)^{a+3}}\right\}x^a.$$

We have, then,

$$\bar{R}^2 = \frac{(1-\rho^2)^{a+2}}{(\rho^2)^{a+1}} \int_0^{\rho^2} x\, d\left\{\frac{(a+1+x)(1-x/\rho^2)x^a}{(1-x)^{a+3}} + \frac{x^{a+1}/\rho^2}{(1-x)^{a+2}}\right\}$$

$$= 1 - \frac{(1-\rho^2)^{a+2}}{(\rho^2)^{a+1}} \int_0^{\rho^2} \frac{(a+1+x)(1-x/\rho^2)x^a}{(1-x)^{a+3}}\, dx - \left(\frac{1-\rho^2}{\rho^2}\right)^{a+2} \int_0^{\rho^2} \frac{x^{a+1}}{(1-x)^{a+2}}\, dx,$$

on integrating by parts,

$$= 1 - \frac{(1-\rho^2)^{a+2}}{(\rho^2)^{a+1}} \int_0^{\rho^2} (1-x/\rho^2)\, d\left\{\frac{x^{a+1}}{(1-x)^{a+2}}\right\} - \left(\frac{1-\rho^2}{\rho^2}\right)^{a+2} \int_0^{\rho^2} \frac{x^{a+1}}{(1-x)^{a+2}}\, dx,$$

from the integral of (4) on replacing a by $a+1$,

$$= 1 - 2\left(\frac{1-\rho^2}{\rho^2}\right)^{a+2} \int_0^{\rho^2} \frac{x^{a+1}}{(1-x)^{a+2}}\, dx \quad \text{.....(7),}$$

on further integrating by parts.

CASE 3. $b=3$. The distribution is

$$df = \frac{(1-\rho^2)^{a+3}}{2(\rho^2)^a}$$

$$\times \frac{\{a(a+1)(a+2)+9(a+1)(a+2)x+18(a+2)x^2+6x^3\}}{(1-x)^{a+6}} x^{a-1}(1-x/\rho^2)^2\, dx \quad (8),$$

and the indefinite integral

$$\frac{(1-\rho^2)^{a+3}}{2(\rho^2)^a}\left\{\frac{(a+3)(a+4)(1-x/\rho^2)^2}{(1-x)^{a+5}} - \frac{2(a+3)(2-3x/\rho^2)(1-x/\rho^2)}{(1-x)^{a+4}} + \frac{2(1-3x/\rho^2+3x^2/\rho^4)}{(1-x)^{a+3}}\right\} x^a.$$

This may be written

$$\frac{(1-\rho^2)^{a+3}}{2(\rho^2)^a}\left[\frac{\{(a+1)(a+2)+4(a+2)x+2x^2\}(1-x/\rho^2)^2}{(1-x)^{a+5}} + \frac{2x/\rho^2(a+2+x)(1-x/\rho^2)}{(1-x)^{a+4}} + \frac{2x^2/\rho^4}{(1-x)^{a+3}}\right] x^a.$$

For the mean value of R^2 we have, on integrating by parts,

$$\bar{R}^2 = 1 - \frac{(1-\rho^2)^{a+3}}{2(\rho^2)^{a+1}} \int_0^{\rho^2} \frac{\{(a+1)(a+2)+4(a+2)x+2x^2\}x^a(1-x/\rho^2)^2}{(1-x)^{a+5}}\, dx$$

$$- \frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+2}} \int_0^{\rho^2} \frac{(a+2+x)x^{a+1}(1-x/\rho^2)}{(1-x)^{a+4}}\, dx - \left(\frac{1-\rho^2}{\rho^2}\right)^{a+3} \int_0^{\rho^2} \frac{x^{a+2}}{(1-x)^{a+3}}\, dx.$$

Now utilising the integral of (6) and replacing a by $a+1$, we may write the first of these integrals in the form

$$\frac{(1-\rho^2)^{a+3}}{2(\rho^2)^{a+1}} \int_0^{\rho^2} \left\{\frac{(a+2+x)(1-x/\rho^2)x^{a+1}}{(1-x)^{a+4}} + \frac{x^{a+2}/\rho^2}{(1-x)^{a+3}}\right\}$$

$$= \frac{(1-\rho^2)^{a+3}}{2(\rho^2)^{a+2}} \int_0^{\rho^2} \frac{+x)(1-x/\rho^2)x^{a+1}}{(1-x)^{a+4}}\, dx + \frac{1}{2}\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3} \int_0^{\rho^2} \frac{x^{a+2}}{(1-x)^{a+3}}\, dx,$$

on integrating by parts. It follows that

$$\bar{R}^2 = 1 - \frac{3}{2}\frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+2}} \int_0^{\rho^2} \frac{(a+2+x)x^{a+1}(1-x/\rho^2)}{(1-x)^{a+4}}\, dx - \frac{3}{2}\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3} \int_0^{\rho^2} \frac{x^{a+2}}{(1-x)^{a+3}}\, dx.$$

The reduced integral may be simplified by using the integral of (4), in which a is now replaced by $a+2$, and is

$$\frac{3}{2}\frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+2}}\int_0^{\rho^2}(1-x/\rho^2)\,d\left\{\frac{x^{a+2}}{(1-x)^{a+3}}\right\}=\frac{3}{2}\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3}\int_0^{\rho^2}\frac{x^{a+2}}{(1-x)^{a+3}}dx,$$

on integrating by parts. Finally we have

$$\overline{R}^2=1-3\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3}\int_0^{\rho^2}\frac{x^{a+2}}{(1-x)^{a+3}}dx \quad \text{.....(9).}$$

It is evident from (5), (7) and (9) that the general result for any integral b is

$$\overline{R}^2=1-b\left(\frac{1-\rho^2}{\rho^2}\right)^{a+b}\int_0^{\rho^2}\frac{x^{a+b-1}}{(1-x)^{a+b}}dx \quad \text{.....(10).}$$

Now when x is less than unity the denominator may be expanded in a convergent series, which when integrated term by term yields the result

$$\begin{aligned}\int_0^{\rho^2}\frac{x^{a+b-1}}{(1-x)^{a+b}}dx&=\left[\frac{x^{a+b}}{a+b}\left\{1+\frac{(a+b)^2}{a+b+1}x+\frac{(a+b)^2(a+b+1)^2}{2\,!\,(a+b+1)(a+b+2)}x^2+\ldots\right\}\right]_0^{\rho^2}\\&=\frac{(\rho^2)^{a+b}}{a+b}F(a+b,\,a+b,\,a+b+1,\,\rho^2)\\&=\frac{1}{a+b}\frac{(\rho^2)^{a+b}}{(1-\rho^2)^{a+b-1}}F(1,\,1,\,a+b+1,\,\rho^2) \quad \text{.....(11),}\end{aligned}$$

using the Euler transformation of the hypergeometric series. The series in (11) is absolutely convergent even for $\rho^2=1$, since $a+b-1$ is always positive*. We therefore have

$$\overline{R}^2=1-\frac{b}{a+b}(1-\rho^2)F(1,\,1,\,a+b+1,\,\rho^2) \quad \text{.....(12)}$$

as our final form. Since x (or ρ^2) may take the value unity in the limiting case, some consideration is necessary as to the validity of the solution we have reached for the integral in (10), where the integrand may become infinite. In this case the important part of the integral is the denominator, and we have

$$b\left(\frac{1-\rho^2}{\rho^2}\right)^{a+b}\int_0^{\rho^2}\frac{\text{const.}\,dx}{(1-x)^{a+b}}=\text{const.}\left(\frac{1-\rho^2}{\rho^2}\right)^{a+b}\left[\frac{1}{(1-x)^{a+b-1}}\right]_0^{\rho^2}\to 0 \text{ as } \rho^2\to 1.$$

Equation (12) therefore gives the mean value of R^2 valid over the whole range of ρ^2 from 0 to 1. We note that for $\rho=0$ we have

$$\overline{R}^2=\frac{a}{a+b},$$

agreeing with Fisher's result† from the simplified distribution (1). Also for $\rho^2=1$ we have $\overline{R}^2=1$.

For comparison with our exact result (12) we have Hall's approximate value‡, which in our notation is

$$\overline{R}^2=\rho^2+\frac{(1-\rho^2)(a-\rho^2)}{a+b+\frac{1}{2}}=\frac{a+(b-\frac{1}{2})\rho^2+\rho^4}{a+b+\frac{1}{2}} \quad \text{.....(13).}$$

* See Whittaker and Watson, *Modern Analysis*, p. 25.

† R. A. Fisher, *Phil. Trans.* B, Vol. 213, 1924, p. 92.

‡ P. Hall, *loc. cit.*

Now (12) can be written

$$\frac{a}{a+b}+\frac{b}{a+b+1}\rho^2 F(1,\,1,\,a+b+2,\,\rho^2)$$

$$=\frac{a}{a+b}+\frac{b}{a+b+1}\rho^2\left\{1+\frac{1}{a+b+2}\rho^2+\frac{1.2}{(a+b+2)(a+b+3)}\rho^4+\ldots\right\} \quad \ldots(14).$$

It is evident that the approximation in (13) consists in supposing that for large N (which is equal to $2(a+b+\frac{1}{2})$ in our notation) $a+b$ and $a+b+1$ may be safely replaced by $a+b+\frac{1}{2}$, while b is replaced by $b-\frac{1}{2}$ and $b/(a+b+2)$ is replaced by unity, and terms of higher order are neglected. To give a numerical example, suppose there are 6 independent variates and the sample is of size 101. Then $a=3$, $b=47$. If we take $\rho^2=0$, 0·5 and 1, we find $\overline{R}^2=0{\cdot}0594$, 0·5248 and 1 respectively from (13), and 0·06 (the correct result), 0·5252 (correct result 0·5253) and 0·9993 from the first three terms only of (14). (We would naturally, however, use (12) for preference for ρ^2 nearly equal to 1; for $\rho^2=1$ exactly we get the correct result from (12), or by using the well-known formula for the sum of the hypergeometric in (14).) It would appear, therefore, to be desirable to improve the approximate formula (13), as it gives an underestimate of the correct mean value, and we would suggest the use of the first three terms of (14), except when ρ^2 is large, when formula (12), using the first two or three terms of the hypergeometric series, should be used, e.g. for $\rho^2=0{\cdot}9$ formula (13) gives $\overline{R}^2=0{\cdot}90416$, while two terms of the hypergeometric in (12) give 0·90434 and three terms 0·90428, which is correct to the last place shown.

It may be mentioned that the value derived for $\overline{R}^2$ by Fisher in 1924* by averaging the numerator and denominator of the expression for R^2, and which may be written

$$\overline{R}^2=1-\frac{b}{a+b}(1-\rho^2),$$

differs from the exact value (12) by a term involving $(F-1)$, of the order of $1/N$.

B. *Determination of Second Moment of R^2.*

This involves a repetition of the procedure we have gone through for determining the mean value. The second moment about zero is obtained by multiplying the distribution by R^4, i.e. by x^2/ρ^4, and integrating for x from 0 to ρ^2.

CASE 1. $b=1$.

We have

$$\mu_2'(R^2)=\frac{(1-\rho^2)^{a+1}}{(\rho^2)^{a+2}}\int_0^{\rho^2}x^2 d\left(\frac{x^a}{(1-x)^{a+1}}\right)=1-\frac{2(1-\rho^2)^{a+1}}{(\rho^2)^{a+2}}\int_0^{\rho^2}\frac{x^{a+1}}{(1-x)^{a+1}}dx,$$

on integrating by parts. For comparison with the other cases we shall leave this result meantime in the form

$$1+\left(\frac{1-\rho^2}{\rho^2}\right)^{a+1}\int_0^{\rho^2}\frac{x^a(0-2x/\rho^2)}{(1-x)^{a+1}}dx \quad \ldots\ldots\ldots\ldots\ldots\ldots(15).$$

* R. A. Fisher, *loc. cit.* p. 92.

CASE 2. $b=2$.

From A, Case 2,

$$\mu_2'(R^2)=\left(\frac{1-\rho^2}{\rho^2}\right)^{a+2}\int_0^{\rho^2}x^2\,d\left\{\frac{(a+1+x)(1-x/\rho^2)x^a}{(1-x)^{a+3}}+\frac{x^{a+1}/\rho^2}{(1-x)^{a+2}}\right\}$$

$$=1-2\left(\frac{1-\rho^2}{\rho^2}\right)^{a+2}\int_0^{\rho^2}\frac{(a+1+x)\,x^{a+1}(1-x/\rho^2)}{(1-x)^{a+3}}\,dx-2\,\frac{(1-\rho^2)^{a+2}}{(\rho^2)^{a+3}}\int_0^{\rho^2}\frac{x^{a+2}}{(1-x)^{a+2}}\,dx$$

$$=1-2\left(\frac{1-\rho^2}{\rho^2}\right)^{a+2}\int_0^{\rho^2}x\,(1-x/\rho^2)\,d\left\{\frac{x^{a+1}}{(1-x)^{a+2}}\right\}-2\,\frac{(1-\rho^2)^{a+2}}{(\rho^2)^{a+3}}\int_0^{\rho^2}\frac{x^{a+2}}{(1-x)^{a+2}}\,dx$$

$$=1+2\left(\frac{1-\rho^2}{\rho^2}\right)^{a+2}\int_0^{\rho^2}\frac{x^{a+1}(1-3x/\rho^2)}{(1-x)^{a+2}}\,dx \qquad \text{.....(16).}$$

CASE 3. $b=3$.

From A, Case 3, we have

$$\mu_2'(R^2)=\frac{(1-\rho^2)^{a+3}}{2\,(\rho^2)^{a+2}}\int_0^{\rho^2}x^2\,d\left[\frac{\{(a+1)(a+2)+4(a+2)x+2x^2\}(1-x/\rho^2)^2x^a}{(1-x)^{a+5}}\right.$$
$$\left.+\frac{2x^{a+1}(a+2+x)(1-x/\rho^2)}{\rho^2(1-x)^{a+4}}+\frac{2x^{a+2}}{\rho^4(1-x)^{a+3}}\right]$$

$$=1-\frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+2}}\int_0^{\rho^2}\frac{\{(a+1)(a+2)+4(a+2)x+2x^2\}\,x^{a+1}(1-x/\rho^2)^2}{(1-x)^{a+5}}\,dx$$
$$-2\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3}\int_0^{\rho^2}\frac{(a+2+x)x^{a+2}(1-x/\rho^2)}{(1-x)^{a+4}}\,dx-2\,\frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+4}}\int_0^{\rho^2}\frac{x^{a+3}}{(1-x)^{a+3}}\,dx$$

$$=1-\frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+2}}\int_0^{\rho^2}x\,(1-x/\rho^2)\,d\left\{\frac{(a+2+x)(1-x/\rho^2)x^{a+1}}{(1-x)^{a+4}}+\frac{x^{a+2}/\rho^2}{(1-x)^{a+3}}\right\}$$
$$-2\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3}\int_0^{\rho^2}\frac{(a+2+x)x^{a+2}(1-x/\rho^2)}{(1-x)^{a+4}}\,dx-2\,\frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+4}}\int_0^{\rho^2}\frac{x^{a+3}}{(1-x)^{a+3}}\,dx$$

$$=1+\frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+2}}\int_0^{\rho^2}\frac{(a+2+x)\,x^{a+1}(1-x/\rho^2)(1-4x/\rho^2)}{(1-x)^{a+4}}\,dx$$
$$+\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3}\int_0^{\rho^2}\frac{x^{a+2}(1-4x/\rho^2)}{(1-x)^{a+3}}\,dx$$

$$=1+\frac{(1-\rho^2)^{a+3}}{(\rho^2)^{a+2}}\int_0^{\rho^2}(1-x/\rho^2)\,(1-4x/\rho^2)\,d\left\{\frac{x^{a+2}}{(1-x)^{a+3}}\right\}$$
$$+\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3}\int_0^{\rho^2}\frac{x^{a+2}(1-4x/\rho^2)}{(1-x)^{a+3}}\,dx$$

$$=1+3\left(\frac{1-\rho^2}{\rho^2}\right)^{a+3}\int_0^{\rho^2}\frac{x^{a+2}(2-4x/\rho^2)}{(1-x)^{a+3}}\,dx \qquad \text{.....(17).}$$

From (15), (16) and (17) it appears that the general result for any integral b is

$$\mu_2'(R^2)=1+b\left(\frac{1-\rho^2}{\rho^2}\right)^{a+b}\int_0^{\rho^2}\frac{x^{a+b-1}\{b-1-(b+1)\,x/\rho^2\}}{(1-x)^{a+b}}\,dx.$$

Now we already have

$$\left(\frac{1-\rho^2}{\rho^2}\right)^{a+b}\int_0^{\rho^2}\frac{x^{a+b-1}}{(1-x)^{a+b}}\,dx=\frac{(1-\rho^2)}{a+b}F\,(1,1,a+b+1,\rho^2),$$

while by a similar expansion of the denominator and integration term by term we have

$$\frac{(1-\rho^2)^{a+b}}{(\rho^2)^{a+b+1}}\int_0^{\rho^2}\frac{x^{a+b}}{(1-x)^{a+b}}\,dx=\frac{(1-\rho^2)}{a+b+1}F(2,1,a+b+2,\rho^2) \quad \text{.....(18).}$$

The same considerations as were examined in the deduction of (11) show that this integration is valid even when $\rho^2 \to 1$, and the hypergeometric series in (18) is also absolutely convergent even for $\rho^2=1$. We then have, for the second moment of R^2 about zero,

$$\mu_2'(R^2)=1+\frac{b(b-1)}{a+b}(1-\rho^2)F(1,1,a+b+1,\rho^2)$$
$$-\frac{b(b+1)}{a+b+1}(1-\rho^2)F(2,1,a+b+2,\rho^2) \quad \text{.....(19).}$$

For $\rho^2=0$ this becomes

$$\mu_2'(R^2)=\frac{a(a+1)}{(a+b)(a+b+1)},$$

which may be derived directly from the distribution (1), while for $\rho^2=1$ we have

$$\mu_2'(R^2)=1.$$

The second moment about the mean, or variance, of R^2, may now be obtained, for, since

$$\overline{R}^2=1-\frac{b}{a+b}(1-\rho^2)F(1,1,a+b+1,\rho^2),$$

we have

$$\sigma^2_{R^2}=\mu_2'(R^2)-(\overline{R}^2)^2$$
$$=b(b+1)(1-\rho^2)\left\{\frac{1}{a+b}F(1,1,a+b+1,\rho^2)-\frac{1}{a+b+1}F(2,1,a+b+2,\rho^2)\right\}$$
$$-\frac{b^2}{(a+b)^2}(1-\rho^2)^2F^2(1,1,a+b+1,\rho^2),$$

and this, on reduction, is equal to

$$\frac{b(b+1)(1-\rho^2)^2}{(a+b)(a+b+1)}F(2,2,a+b+2,\rho^2)-\frac{b^2(1-\rho^2)^2}{(a+b)^2}F^2(1,1,a+b+1,\rho^2)\ldots(20).$$

An alternative form for this expression is

$$\frac{b(b+1)}{a+b}(1-\rho^2)^2\frac{dF}{d(\rho^2)}-(1-\overline{R}^2)^2 \quad \text{.....(21),}$$

where F stands for the hypergeometric series $F(1,1,a+b+1,\rho^2)$, i.e. the same series which occurs in the expression (12) for the mean value of R^2. This series is the only part of our results (12) and (21) which is at all difficult to calculate, although for values of ρ^2 up to 0·5 and a reasonably large N a very few terms of the series should suffice. A table of the series F for values of $a+b+1$ proceeding by half-integers, and for a number of values of ρ^2, with, possibly, a table of its derivative, would be useful in this connection.

The only result known hitherto for the variance of R^2 is the approximate one of P. Hall (*loc. cit.*):

$$\sigma^2_{R^2}=4\rho^2(1-\rho^2)^2/N=2\rho^2(1-\rho^2)^2/(a+b+\tfrac{1}{2}) \quad \text{.....(22).}$$

This result is correct to terms of the order of $1/N$, but its weakness lies in the fact that for $\rho^2 \to 0$ it gives $\sigma^2_{R^2} \to 0$, whereas in fact we know from the distribution (1) directly*, or from (20) on putting $\rho^2 = 0$, that

$$\text{For } \rho^2 = 0, \qquad \sigma^2_{R^2} = \frac{ab}{(a+b)^2(a+b+1)}.$$

This result is of the order of $1/N^2$, which explains wherein the approximation (22) is insufficient†. The terms in $1/N$ involve ρ^2, and when this is equal to zero the terms all vanish, while the part that does not vanish with ρ^2 is not given, being of the order of $1/N^2$. An exact formula is always to be preferred to an approximate one, proceeding in powers of $1/N$. If N is not really large the first term or two will not be adequate to give precision enough; while a more serious objection, illustrated in the case before us, is that in particular cases the early terms of a series may vanish, and the first term of importance may be a term neglected.

The nature of the approximation in (22) may be seen from (20) on expanding the hypergeometric series as far as the terms in ρ^2. The parts outside involving a and b are nearly unity for large N. If we count them as unity and replace the $a+b+2$ and $a+b+1$ of the hypergeometric series by $\frac{1}{2}N$ we find, approximately,

$$\sigma^2_{R^2} = (1-\rho^2)^2\left[1 + \frac{4}{\frac{1}{2}N}\rho^2 - 1 - \frac{2}{\frac{1}{2}N}\rho^2\right] = 4\rho^2(1-\rho^2)^2/N,$$

as in (22).

As a numerical example let $a = 3$, $b = 47$, and $\rho^2 = 0{\cdot}5$. From (22) we have $\sigma^2_{R^2} = 0{\cdot}00495$. The correct result, from (20), is $0{\cdot}0047241$. A much better approximation is obtained by retaining the exact values of the parts outside the hypergeometric series in (20) and calculating the series up to terms in ρ^4. This yields $0{\cdot}00470$. We have chosen the case of $N = 101$ for the purposes of illustration, and even here the approximate forms (13) and (22) are not good enough. For smaller samples the discrepancy will be even wider, and it is obvious that the exact forms (12) and (20) must be used in such cases to secure reliable results.

C. *Mean and Second Moment of R.*

We have dealt so far with R^2, as having a rather simpler sampling distribution than R. We know that the mean value of R, for the special case of no correlation in the population, is of the form‡

$$\overline{R} = \frac{(a-\frac{1}{2})\,!\,(a+b-1)\,!}{(a-1)\,!\,(a+b-\frac{1}{2})\,!} \qquad \text{.....(23),}$$

where $x!$ is written for the factorial function, or $\Gamma(x+1)$, even when x is not an integer. It is hardly to be expected, therefore, that the more general form for any ρ should be simple. A similar method of attack to that in A does, in fact, lead to a solution for the special cases $b = 1$, 2 and 3, and it is seen that the mean value will in the general case involve a number of hypergeometric series equal in number to b,

* J. Wishart, *Mem. Roy. Met. Soc.* Vol. xi. No. 13, 1928, p. 34.

† Indicated on other grounds by P. Hall, *loc. cit.* p. 108.

‡ J. Wishart, *loc. cit.* p. 34; P. Hall, *loc. cit.* p. 109.

but it does not appear that the expressions are capable of any very great degree of simplification. The first three results are

$$b=1.\quad \overline{R}=1-\frac{1-\rho^2}{2a+1}F(\tfrac{1}{2},1,a+\tfrac{3}{2},\rho^2);$$

$$b=2.\quad \overline{R}=1-\frac{1}{2}\frac{1-\rho^2}{2a+1}F(-\tfrac{1}{2},1,a+\tfrac{3}{2},\rho^2)-\frac{3}{2}\frac{1-\rho^2}{2a+3}F(\tfrac{1}{2},1,a+\tfrac{5}{2},\rho^2);$$

$$b=3.\quad \overline{R}=1-\frac{3}{8}\frac{1-\rho^2}{2a+1}F(-\tfrac{3}{2},1,a+\tfrac{3}{2},\rho^2)-\frac{3}{4}\frac{1-\rho^2}{2a+3}F(-\tfrac{1}{2},1,a+\tfrac{5}{2},\rho^2)$$
$$-\frac{15}{8}\frac{1-\rho^2}{2a+5}F(\tfrac{1}{2},1,a+\tfrac{7}{2},\rho^2).$$

In view of this difficulty, and also bearing in mind that R^2 is calculated first before extracting the square root, it would seem desirable to apply the usual tests of significance to R^2, and not to R. The second moment of R about zero is, of course, identical with the mean value of R^2, i.e. our formula (12), and we therefore have

$$\sigma^2_R=1-\frac{b}{a+b}(1-\rho^2)F(1,1,a+b+1,\rho^2)-(\overline{R})^2 \quad \text{.....(24).}$$

Further than this it is hardly practicable to proceed. What we have done in this section will illustrate the difficulties experienced by other authors* in obtaining approximate expressions for the mean value of R.

* L. Isserlis, *Phil. Mag.* Vol. XXXIV. 1917, pp. 205—220; P. Hall, *loc. cit.* pp. 108—109.

APPENDIX TO A PAPER BY DR WISHART.

Tables of the Mean Value and Squared Standard Deviation of the Square of a Multiple Correlation Coefficient.

EDITORIAL.

Dr Wishart has provided in his paper the formulae giving the Mean Value, $\overline{R}^2$, and the Squared Standard Deviation $\sigma^2_{R^2}$ of the square of a multiple correlation coefficient R. Let us suppose N = size of sample and n = *total* number of variates*, then Dr Wishart's formulae may be expressed as follows:

$$\overline{R}^2 = 1 - \frac{N-n}{N-1}(1-\rho^2)\,F(1, 1, \tfrac{1}{2}(N+1), \rho^2) \quad \text{.....(i),}$$

$$\sigma^2_{R^2} = \frac{(N-n)(N-n+2)}{(N-1)(N+1)}(1-\rho^2)^2\,F(2, 2, \tfrac{1}{2}(N+3), \rho^2) - (1-\overline{R}^2)^2 \quad \text{...(ii),}$$

where F is the hypergeometrical function. We may write these as follows:

$$\overline{R}^2 = 1 - \frac{N-n}{N-2}\gamma_1 \quad \text{.....(i) } \textit{bis,}$$

$$\sigma^2_{R^2} = \frac{(N-n)(N-n+2)}{(N-2)N}\gamma_2 - (1-\overline{R}^2)^2 \quad \text{.....(ii) } \textit{bis.}$$

The only parts of these formulae which involve n, the total number of variates, are the coefficients of γ_1 and γ_2; these change with the order $n-1$ of the multiple correlation coefficient.

Now $\gamma_1 = \mu_2' - 1$ and $\gamma_4 = \mu_4' - 2\mu_2' + 1$, of the paper in *Biometrika*, Vol. XI. pp. 334—335, and although the numerical values of μ_2' and μ_4' are not given in the Tables attached to it, they exist in the Archives of the Laboratory on the working sheets from which the μ_2 and μ_4 of the frequency distributions of r were obtained. It is therefore only a matter of picking out of those sheets the values of μ_2' and μ_4' and so finding γ_1 and γ_2. This has been done and Tables I and II below provide their values.

For samples of 3† to 25, there is no need of interpolation for N; we require only to interpolate for ρ. For most practical purposes central difference interpolation to δ^2 will suffice. Beyond 25, the two adjacent values of γ_1 and γ_2 are so close that linear interpolation for N will as a rule be adequate.

Illustration (i). Let us take Dr Wishart's example $N = 101$, $n = 7$, and $\rho^2 = \cdot5$, or $\rho = \cdot7071$ nearly. This lies between $\rho = \cdot7$ and $\cdot8$.

* Wishart puts $a = \frac{1}{2}$ (number of *independent* variates) $= \frac{1}{2}$ (Fisher's n_1) $= \frac{1}{2}$ (our $(n-1)$) and $b = \frac{1}{2}$ (size of sample – *total* number of variates) $= \frac{1}{2}$ (Fisher's n_2) $= \frac{1}{2}$ (our $N-n$).

† For $N=2$, we can only take $n=2$, i.e. ordinary correlation and then $\overline{R}^2=1$ and $\sigma^2_{R^2}=0$.

First to find γ_1 using Everett's Central Difference formula we have for $N=100$, $\theta=\cdot071$, $\phi=\cdot929$,

$z_0=\cdot509{,}8429$, $z_1=\cdot360{,}9965$, and $\delta^2 z_0=-\cdot020{,}5732$, $\delta^2 z_1=-\cdot020{,}9530$.

Hence:

$$\begin{aligned} z_\theta &= \cdot929\times\cdot509{,}8429+\cdot071\times\cdot360{,}9965 \\ &\qquad -\tfrac{1}{6}(\cdot929\times\cdot071)\{1\cdot929(-\cdot020{,}5732)+1\cdot071(-\cdot020{,}9530)\} \\ &= \cdot499{,}2748+\cdot010{,}993\times\cdot062{,}1264 \\ &= \cdot499{,}9578=\gamma_1 \text{ for } N=100. \end{aligned}$$

Similarly for $N=200$,

$$\begin{aligned} z_\theta &= \cdot929\times\cdot509{,}9355+\cdot071\times\cdot360{,}5013 \\ &\qquad -\cdot010{,}993\{1\cdot929\times(-\cdot020{,}2884)+1\cdot071\times(-\cdot020{,}4733)\} \\ &= \cdot499{,}9970=\gamma_1 \text{ for } N=200. \end{aligned}$$

Clearly linear interpolation for $N=101$ will suffice and we have

$$\gamma_1=\cdot499{,}9582 \text{ for } N=101.$$

Thus
$$\begin{aligned} \bar{R}^2 &= 1-\frac{N-n}{N-2}\gamma_1=1-\frac{94}{99}\times\cdot499{,}9582 \\ &= \cdot525{,}2922, \end{aligned}$$

agreeing completely with Dr Wishart's

$$\bar{R}^2=\cdot5253.$$

We turn now to Table II to find γ_2. We have for $N=100$,

$$z_0=\cdot265{,}0540, \qquad z_1=\cdot133{,}7050,$$
$$\delta^2 z_0=+\cdot016{,}6294, \quad \delta^2 z_1=+\cdot035{,}4198.$$

The values of θ and ϕ are as before. Hence

$$\begin{aligned} z_\theta &= \cdot929\times\cdot265{,}0540+\cdot071\times\cdot133{,}7050 \\ &\qquad -\cdot010{,}993\{1\cdot929\times\cdot016{,}6294+1\cdot071\times\cdot035{,}4198\} \\ &= \cdot254{,}9586=\gamma_2 \text{ for } N=100. \end{aligned}$$

Similarly

$$\begin{aligned} z_\theta &= \cdot929\times\cdot262{,}5877+\cdot071\times\cdot131{,}6374 \\ &\qquad -\cdot010{,}993\{1\cdot929\times\cdot017{,}8219+1\cdot071\times\cdot036{,}2360\} \\ &= \cdot252{,}4857=\gamma_2 \text{ for } N=200. \end{aligned}$$

Interpolating linearly for $N=101$

$$\begin{aligned} \gamma_2 &= \cdot254{,}9586-\tfrac{1}{100}(\cdot254{,}9586-\cdot252{,}4857) \\ &= \cdot254{,}9586-\cdot000{,}0247 \\ &= \cdot254{,}9339. \end{aligned}$$

Thus
$$\begin{aligned} \sigma^2_{R^2} &= \tfrac{94}{99}\times\tfrac{96}{101}\times\cdot254{,}9339-(\cdot474{,}7078)^2 \\ &= \cdot2300{,}7556-\cdot2253{,}4750 \\ &= \cdot0047{,}2806, \end{aligned}$$

and accordingly $\sigma_{R^2}=\cdot0688$.

TABLE I. *Values of* γ_1.

ρ = Multiple Correlation Coefficient in Parent Population.

N	$\rho=0$	$\rho=\cdot1$	$\rho=\cdot2$	$\rho=\cdot3$	$\rho=\cdot4$	$\rho=\cdot5$	$\rho=\cdot6$	$\rho=\cdot7$	$\rho=\cdot8$	$\rho=\cdot9$	$\rho=1\cdot0$
3	·500 0000^e	·497 4916	·489 8639	·476 7929	·457 6776	·431 5231	·396 6996	·350 4140	·287 3394	·194 7771	$\gamma_1 = 0$ throughout
4	·666 6667	·662 6552	·650 4805$_-$	·629 7020	·599 5127	·558 6019	·504 8836	·434 9287	·342 5947	·214 7098	
5	·750 0000^e	·744 9874	·729 7967	·703 9493	·666 5772	·616 2920	·550 9352	·467 0746	·358 8647	·214 7865$^-$	
6	·800 0000	·794 2730	·776 9367	·747 5077	·705 1162	·648 3886	·575 2286	·482 3934	·364 5810	·212 0292	
7	·833 3333	·827 0708	·808 1306	·776 0392	·729 9499	·668 5399	·589 8216	·490 7889	·366 6893	·209 2399	
8	·857 1429	·850 4640	·830 2798	·796 1328	·747 2098	·682 2515$^-$	·599 3905$^-$	·495 8553	·367 3848	·206 8786	
9	·875 0000	·867 9883	·846 8112	·811 0298	·759 8680	·692 1325$^-$	·606 0737	·499 1362	·367 4822	·204 9492	
10	·888 8889	·881 6049	·859 6170	·822 5056	·769 5313	·699 5654	·610 9686	·501 3787	·367 3186	·203 3748	
11	·900 0000	·892 4893	·869 8268	·831 6120	·777 1412	·705 3462	·614 6887	·502 9781	·367 0441	·202 0788	
12	·909 0909	·901 3883	·878 1560	·839 0113	·783 2842	·709 9629	·617 6007	·504 1586	·366 7291	·200 9995$^-$	
13	·916 6667	·908 7997	·885 0795$^+$	·845 1405$^-$	·788 3441	·713 7305$^+$	·619 9357	·505 0551	·366 4072	·200 0899	
14	·923 0769	·915 0675$^-$	·890 9251	·850 2997	·792 5824	·716 8607	·621 8458	·505 7521	·366 0942	·199 3146	
15	·928 5714	·920 4374	·895 9258	·854 7017	·796 1830	·719 5009	·623 4348	·506 3051	·365 7975$^+$	·198 6470	
16	·933 3333	·925 0894	·900 2524	·858 5012	·799 2788	·721 7567	·624 7757	·506 7515$^+$	·365 5199	·198 0666	
17	·937 5000^e	·929 1583	·904 0324	·861 8137	·801 9687	·723 7055$^+$	·625 9214	·507 1173	·365 2619	·197 5579	
18	·941 1765$_-$	·932 7474	·907 3630	·864 7270	·804 3271	·725 4056	·626 9108	·507 4209	·365 0229	·197 1085$_+$	
19	·944 4444	·935 9367	·910 3200	·867 3089	·806 4115$^-$	·726 9012	·627 7733	·507 6759	·364 8018	·196 7089	
20	·947 3684	·938 7895	·912 9627	·869 6130	·808 2668	·728 2269	·628 5313	·507 8923	·364 5972	·196 3513	
21	·950 0000^e	·941 3564	·915 3387	·871 6815$^+$	·809 9288	·729 4099	·629 2029	·508 0776	·364 4077	·196 0294	
22	·952 3810	·943 6783	·917 4864	·873 5489	·811 4260	·730 4719	·629 8016	·508 2377	·364 2320	·195 7384	
23	·954 5455$^-$	·945 7887	·919 4371	·875 2431	·812 7817	·731 4305$^-$	·630 3382	·508 3768	·364 0689	·195 4738	
24	·956 5217	·947 7152	·921 2169	·876 7871	·814 0150$^-$	·732 2999	·630 8217	·508 4988	·363 9171	·195 2325$_-$	
25	·958 3333	·949 4809	·922 8471	·878 1999	·815 1417	·733 0921	·631 2607	·508 6062	·363 7757	·195 0113	
50	·979 5918	·970 1765$_+$	·941 8878	·894 5964	·828 0836	·742 0365$_+$	·636 0401	·509 5673	·361 9643	·192 4322	
100	·989 8990	·980 1941	·951 0569	·902 4191	·834 1662	·746 1358	·638 1161	·509 8429	·360 9965$_-$	·191 1971	
200	·994 9748	·985 1232	·955 5563	·906 2394	·837 1137	·748 0967	·639 0813	·509 9355$^-$	·360 5013	·190 5938	
400	·997 4937	·987 5681	·957 7851	·908 1271	·838 5644	·749 0554	·639 5464	·509 9712	·360 2513	·190 2957	

TABLE II. *Values of* γ_2.

ρ = multiple correlation coefficient in parent population.

N	$\rho=0$	$\rho=\cdot1$	$\rho=\cdot2$	$\rho=\cdot3$	$\rho=\cdot4$	$\rho=\cdot5$	$\rho=\cdot6$	$\rho=\cdot7$	$\rho=\cdot8$	$\rho=\cdot9$	$\rho=1\cdot0$
3	·375 0000^e	·372 4937	·364 8983	·351 9747	·333 2885	·308 1460	·275 4675	·233 5373	·179 4323	·107 3932	$\gamma_2=0$ throughout
4	·533 3333	·528 7543	·514 9245	·491 5596	·458 1615	·413 9809	·357 9592	·288 6614	·204 2769	·103 3340	
5	·625 0000^e	·618 7437	·599 9000	·568 2444	·523 4070	·464 8896	·392 1267	·304 6799	·202 9060	·090 7534	
6	·685 7143	·678 0919	·655 1842	·616 8793	·563 0261	·493 5091	·408 4240	·308 5062	·196 3093	·080 2621	
7	·729 1667	·720 4167	·694 1694	·650 4492	·589 3602	·511 2142	·416 8017	·307 9994	·189 2911	·072 4402	
8	·761 9048	·752 2115	·723 1804	·674 9803	·607 9805	·522 9222	·421 2652	·305 9241	·182 9972	·066 6492	
9	·787 5000^e	·777 0076	·745 6250	·693 6651	·621 7588	·531 0590	·423 6466	·303 3544	·177 6088	·062 2889	
10	·808 0808	·796 9032	·763 5103	·708 3551	·632 3174	·536 9375	·424 8626	·300 7280	·173 0497	·058 9315	
11	·825 0000^e	·813 2292	·778 0992	·720 1968	·640 6374	·541 3193	·425 4021	·298 2218	·169 1896	·056 2887	
12	·839 1608	·826 8713	·790 2266	·729 9390	·647 3439	·544 6706	·425 5401	·295 9003	·165 9037	·054 1657	
13	·851 1905	·838 4441	·800 4668	·738 0896	·652 8528	·547 2896	·425 4371	·293 7793	·163 0858	·052 4291	
14	·861 5384	·848 3864	·809 2288	·745 0063	·657 4508	·549 3742	·425 1897	·291 8522	·160 6501	·050 9860	
15	·870 5357	·857 0213	·816 8104	·750 9476	·661 3413	·551 0599	·424 8575	·290 1046	·158 5286	·049 7700	
16	·878 4314	·864 5913	·823 4353	·756 1045	·664 6719	·552 4422	·424 4778	·288 5192	·156 6673	·048 7326	
17	·885 4167	·871 2823	·829 2736	·760 6217	·667 5529	·553 5893	·424 0747	·287 0785	·155 0230	·047 8384	
18	·891 6409	·877 2395	·834 4572	·764 6106	·670 0676	·554 5517	·423 6630	·285 7662	·153 5612	·047 0598	
19	·897 2221	·882 5772	·839 0908	·768 1579	·672 2801	·555 3668	·423 2526	·284 5679	·152 2541	·046 3765	
20	·902 2556	·887 3877	·843 2571	·771 3331	·674 2408	·556 0630	·422 8491	·283 4706	·151 0788	·045 7721	
21	·906 8182	·891 7453	·847 0233	·774 1910	·675 9898	·556 6624	·422 4575	·282 4630	·150 0169	·045 2338	
22	·910 9731	·895 7098	·850 4445	·776 7769	·677 5588	·557 1820	·422 0789	·281 5353	·149 0532	·044 7517	
23	·914 7728	·899 3363	·853 5659	·779 1278	·678 9738	·557 6355	·421 7147	·280 6785	·148 1749	·044 3171	
24	·918 2608	·902 6624	·856 4254	·781 2741	·680 2560	·558 0334	·421 3656	·279 8856	·147 3711	·043 9240	
25	·921 4743	·905 7253	·859 0544	·783 2410	·681 4230	·558 3846	·421 0318	·279 1496	·146 6331	·043 5661	
50	·960 3841	·942 6956	·890 4692	·806 2600	·694 4546	·561 4768	·416 1002	·269 9046	·137 9239	·039 5739	
100	·980 0980	·961 3408	·906 0742	·817 3368	·700 2710	·562 2454	·413 0324	·265 0540	·133 7050$^-$	·037 7758	
200	·990 0248	·970 7058	·913 8476	·822 7588	·702 9974	·562 4365$^+$	·411 3599	·262 5877	·131 6374	·036 9231	
400	·995 0062	·975 3992	·917 7265$^+$	·825 4397	·704 3143	·562 4842	·410 4906	·261 3463	·130 6148	·036 5079	

Illustration (ii). Let $\rho = \cdot 3$, $N = 315$, $n = 5$. We require to find $\bar{R}^2$ and σ_{R^2}.

This is about as unfavourable an example as we can take for the Tables. It is easier to expand Dr Wishart's Hypergeometrical

$$\bar{R}^2 = 1 - \frac{N-n}{N-1}(1-\rho^2)\,F(1, 1, \tfrac{1}{2}(N+1), \rho^2),$$

which gives $\bar{R}^2 = \cdot 10108$, in the present case. We will, however, compute $\bar{R}^2$ by aid of Table I. The value $N = 315$ occurs in a part of that table, where the argument does not run by equal intervals but logarithmically, i.e.

$$\begin{aligned}
\log\ 25 &= \log 25 + 0 \times \log 2,\\
\log\ 50 &= \log 25 + 1 \times \log 2,\\
\log 100 &= \log 25 + 2 \times \log 2,\\
\log 200 &= \log 25 + 3 \times \log 2,\\
\log 400 &= \log 25 + 4 \times \log 2;
\end{aligned}$$

and we need

$$\begin{aligned}
\log 315 &= \log 25 + \left(\frac{\log 12{\cdot}6}{\log 2}\right)\log 2\\
&= \log 25 + 3{\cdot}655{,}3516 \log 2.
\end{aligned}$$

Hence for equal argument intervals we shall need to interpolate at a distance

$$4 - 3{\cdot}655{,}3516 = \cdot 344{,}6484 = \theta$$

from the 400 value. Write down the terms in reverse order and difference them:

	Δ	Δ^2	Δ^3	Δ^4
·908,1271,				
·906,2394,	− ·001,8877,			
·902,4191,	− ·003,8203,	− ·001,9326,		
·894,5964,	− ·007,8227,	− ·004,0024,	− ·002,0698,	
·878,1999,	− ·016,3965,	− ·008,5738,	− ·004,5714,	− ·002,5016.

The differences are thus slightly diverging, but the forward difference formula will suffice. Accordingly:

$$\begin{aligned}
z_\theta &= \cdot 908{,}1271 - \cdot 344{,}6484 \times \cdot 001{,}8877 + \cdot 112{,}9329 \times \cdot 001{,}9326\\
&\quad - \cdot 062{,}3146 \times \cdot 002{,}0698 + \cdot 041{,}3668 \times \cdot 002{,}5016\\
&= \cdot 908{,}1271 - \cdot 000{,}6506 + \cdot 000{,}2183 - \cdot 000{,}1290 + \cdot 000{,}1035.
\end{aligned}$$

Clearly the required value is greater than ·907,5658 and less than ·907,6693. Taking it as the mean of these we have

$$\begin{aligned}
\gamma_1 &= \cdot 907{,}6175,\\
\bar{R}^2 &= 1 - \tfrac{310}{313} \times \cdot 907{,}6175 = 1 - \cdot 898{,}9183\\
&= \cdot 101{,}0817,
\end{aligned}$$

in excellent agreement with Dr Wishart's result ·10108.

Calculated from the formula

$$\sigma^2_{R^2} = \frac{(N-n)(N-n+2)}{(N-1)(N+1)}(1-\rho^2)^2 F(2, 2, \tfrac{1}{2}(N+3), \rho^2) - (1-\bar{R}^2)^2,$$

we find

$$\sigma_{R^2} = \cdot 03124,$$

again the quicker method.

But it is of interest to see how closely by aid of a logarithmic formula we can get comparable results from a table with apparently absurd stretches of argument. Our differences from Table II are:

	Δ	Δ^2	Δ^3	Δ^4
·825,4397,				
·822,7588,	− ·002,6809,			
·817,3368,	− ·005,4220,	− ·002,7411,		
·806,2600,	− ·011,0768,	− ·005,6548,	− ·002,9137,	
·783,2410,	− ·023,0190,	− ·011,9422,	− ·006,2874,	− ·003,3737.

Accordingly:

$$z_\theta = \cdot 825,4397 - \cdot 344,6484 \times \cdot 002,6809 + \cdot 112,9329 \times \cdot 002,7411 - \cdot 062,3146 \times \cdot 002,9137 + \cdot 041,3668 \times \cdot 003,3737$$
$$= \cdot 825,4397 - \cdot 000,9240 + \cdot 000,3096 - \cdot 000,1816 + \cdot 000,1396.$$

Thus γ_2 lies between ·824,7833 and ·824,6437.

Taking as before the mean of these values we have

$$\gamma_2 = \cdot 824,7135.$$

Using the Equation (ii) *bis*, we have

$$\sigma^2_{R^2} = \tfrac{310}{313} \times \tfrac{312}{315} \times \cdot 824,7135 - (\cdot 898,9183)^2$$
$$= \cdot 000,97566,$$

or
$$\sigma_{R^2} = \cdot 03124,$$

agreeing with the directly computed value.

These results are interesting as showing that by the use of a logarithmic interpolation we may cover by three properly chosen intermediate values the range from 25 to 400, with sufficient accuracy for most statistical purposes.

Table of Normal Curve Functions to each Permille of Frequency.
Computed by T. Kondo, Ph.D., Lond., and revised by Ethel M. Elderton.

$\frac{1}{2}(1+a_x)$	x	z	$\frac{\frac{1}{2}(1+a_x)}{z}$	$\frac{\frac{1}{2}(1-a_x)}{z}$	$\frac{z}{\frac{1}{2}(1+a_x)}$	$\frac{z}{\frac{1}{2}(1-a_x)}$	$\frac{1}{2}(1-a_x)$
·500	·00000 00000	·39894 22804	1·25331 41373	1·25331 41373	·79788 45608	·79788 45608	·500
·501	·00250 66309	·39894 10271	1·25582 47109	1·25081 14386	·79628 94752	·79948 10162	·499
·502	·00501 32775^{5}	·39893 72671	1·25834 32067	1·24831 65676	·79469 57513	·80107 88497	·498
·503	·00751 99557	·39893 10005^{+}	1·26086 96726	1·24582 94777	·79310 33808	·80267 80695^{-}	·497
·504	·01002 66811	·39892 22272	1·26340 41566	1·24335 01223	·79151 23556	·80427 86839	·496
·505	·01253 34696	·39891 09471	1·26594 67072	1·24087 84556	·78992 26676	·80588 07013	·495
·506	·01504 03367	·39889 71602	1·26849 73733	1·23841 44317	·78833 43088	·80748 41300^{5}	·494
·507	·01754 72984	·39888 08664	1·27105 62042	1·23595 80052	·78674 72711	·80908 89786	·493
·508	·02005 43703^{5}	·39886 20656	1·27362 32492	1·23350 91311^{5}	·78516 15465^{-}	·81069 52553	·492
·509	·02256 15684	·39884 07576^{5}	1·27619 85585^{+}	1·23106 77647	·78357 71270	·81230 29687	·491
·510	·02506 89083	·39881 69424	1·27878 21824	1·22863 38615^{+}	·78199 40047^{5}	·81391 21274	·490
·511	·02757 64057^{5}	·39879 06198	1·28137 41714	1·22620 73774	·78041 21718	·81552 27398	·489
·512	·03008 40766	·39876 17895^{+}	1·28397 45769	1·22378 82686	·77883 16202	·81713 48146	·488
·513	·03259 19367	·39873 04516	1·28658 34500	1·22137 64915^{+}	·77725 23422	·81874 83605^{+}	·487
·514	·03510 00018	·39869 66056	1·28920 08429	1·21897 20032	·77567 43300	·82036 33860	·486
·515	·03760 82877	·39866 02515^{-}	1·29182 68076	1·21657 47606	·77409 75757	·82197 99000	·485
·516	·04011 68102	·39862 13890	1·29446 13969	1·21418 47212	·77252 20716	·82359 79111	·484
·517	·04262 55852	·39858 00178	1·29710 46639	1·21180 18427	·77094 78100	·82521 74281	·483
·518	·04513 46285^{+}	·39853 61377	1·29975 66619	1·20942 60831	·76937 47832	·82683 84600	·482
·519	·04764 39560	·39848 97484	1·30241 74450^{-}	1·20705 74008	·76780 29835^{-}	·82846 10154^{5}	·481
·520	·05015 35835^{-}	·39844 08497	1·30508 70673	1·20469 57544	·76623 24032	·83008 51035^{+}	·480
·521	·05266 35269	·39838 94412	1·30776 55836	1·20234 11027	·76466 30348^{5}	·83171 07331	·479
·522	·05517 38022	·39833 55225^{+}	1·31045 30490	1·19999 34050	·76309 48707	·83333 79132	·478
·523	·05768 44251	·39827 90934	1·31314 95191	1·19765 26206	·76152 79033	·83496 66529	·477
·524	·06019 54118	·39822 01535^{5}	1·31585 50498^{5}	1·19531 87094	·75996 21251	·83659 69612	·476
·525	·06270 67780	·39815 87025^{-}	1·31856 96978	1·19299 16313	·75839 75285^{5}	·83822 88473	·475
·526	·06521 85397^{5}	·39809 47399	1·32129 35197	1·19067 13466	·75683 41062	·83986 23204	·474
·527	·06773 07130^{5}	·39802 82653	1·32402 65729	1·18835 78159	·75527 18506	·84149 73896	·473
·528	·07024 33138^{5}	·39795 92783	1·32676 89153	1·18605 10000^{5}	·75371 07543	·84313 40642	·472
·529	·07275 63582	·39788 77785^{-}	1·32952 06050	1·18375 08600^{5}	·75215 08100	·84477 23535^{-}	·471
·530	·07526 98622	·39781 37654	1·33228 17008	1·18145 73573	·75059 20102	·84641 22669	·470
·531	·07778 38417	·39773 72386	1·33505 22618	1·17917 04534	·74903 43477	·84805 38137	·469
·532	·08029 83130	·39765 81976	1·33783 23476	1·17689 01103	·74747 78150^{-}	·84969 70034	·468
·533	·08281 32920	·39757 66418	1·34062 20183^{5}	1·17461 62900	·74592 24049	·85134 18454	·467
·534	·08532 87949^{5}	·39749 25708	1·34342 13346^{5}	1·17234 89549^{5}	·74436 81101^{5}	·85298 83494	·466
·535	·08784 48380	·39740 59840^{5}	1·34623 03576	1·17008 80678	·74281 49235^{-}	·85463 65248	·465
·536	·09036 14372	·39731 68810	1·34904 91487	1·16783 35914	·74126 28376	·85628 63814	·464
·537	·09287 86088^{5}	·39722 52610	1·35187 77700	1·16558 54888	·73971 18454	·85793 79287	·463
·538	·09539 63691	·39713 11235^{+}	1·35471 62841	1·16334 37235^{5}	·73816 19397	·85959 11765^{e}	·462
·539	·09791 47343	·39703 44680	1·35756 47542	1·16110 82591	·73661 31132^{5}	·86124 61346	·461
·540	·10043 37206	·39693 52939	1·36042 32436^{5}	1·15887 90594	·73506 53590	·86290 28127^{5}	·460
·541	·10295 33443	·39683 36004	1·36329 18167	1·15665 60885^{-}	·73351 86698^{5}	·86456 12209	·459
·542	·10547 36218	·39672 93870	1·36617 05379	1·15443 93106^{5}	·73197 30387	·86622 13689	·458
·543	·10799 45695^{-}	·39662 26529	1·36905 94725^{+}	1·15222 86905^{+}	·73042 84584	·86788 32668	·457
·544	·11051 62036	·39651 33976	1·37195 86861^{5}	1·15002 41928	·72888 49220	·86954 69245^{+}	·456
·545	·11303 85407	·39640 16203	1·37486 82450	1·14782 57825^{+}	·72734 24225^{+}	·87121 23523	·455
·546	·11556 15972	·39628 73203	1·37778 82159^{5}	1·14563 34250^{-}	·72580 09529	·87287 95601	·454
·547	·11808 53895^{-}	·39617 04968	1·38071 86662	1·14344 70856	·72426 05061	·87454 85581^{5}	·453
·548	·12060 99342	·39605 11492	1·38365 96638	1·14126 67300	·72272 10753	·87621 93567	·452
·549	·12313 52478	·39592 92767	1·38661 12770	1·13909 23241	·72118 26534	·87789 19661	·451
·550	·12566 13469	·39580 48785^{-}	1·38957 35750^{+}	1·13692 38341	·71964 52336	·87956 63966	·450
·551	·12818 82481^{5}	·39567 79538	1·39254 66273	1·13476 12262	·71810 88090	·88124 26587	·449
·552	·13071 59682	·39554 85017	1·39553 05041	1·13260 44671	·71657 33727	·88292 07628	·448
·553	·13324 45236^{5}	·39541 65215^{+}	1·39852 52762	1·13045 35234	·71503 89178	·88460 07193	·447
·554	·13577 39313	·39528 20124	1·40153 10149	1·12830 83622	·71350 54375^{+}	·88628 25390	·446
·555	·13830 42080	·39514 49734	1·40454 77923	1·12616 89506	·71197 29250^{+}	·88796 62323	·445
·556	·14083 53704	·39500 54037	1·40757 56808	1·12403 52559	·71044 13735^{+}	·88965 18101	·444
·557	·14336 74355^{-}	·39486 3·023^{5}	1·41061 47537	1·12190 72458	·70891 07762	·89133 92829^{5}	·443
·558	·14590 04200^{5}	·39471 86 685^{-}	1·41366 50849	1·11978 48880	·70738 11263	·89302 86618	·442
·559	·14843 43411	·39457 75012	1·41672 67486^{5}	1·11766 81505^{5}	·70585 24172	·89471 99574	·441
·560	·15096 92155^{+}	·394 ,2 17995^{-}	1·41979 98202	1·11555 70016	·70432 46420	·89641 31807	·440

$\frac{1}{2}(1+a_x)$	x	z	$\frac{\frac{1}{2}(1+a_x)}{z}$	$\frac{\frac{1}{2}(1-a_x)}{z}$	$\frac{z}{\frac{1}{2}(1+a_x)}$	$\frac{z}{\frac{1}{2}(1-a_x)}$	$\frac{1}{2}(1-a_x)$
·560	·15096 92155^{+}	·39442 17995^{-}	1·41979 98202	1·11555 70016	·70432 46420	·89641 31807	·440
·561	·15350 50604	·39426 95624	1·42288 43752	1·11345 14094	·70279 77940	·89810 83427	·439
·562	·15604 18928	·39411 47890^{5}	1·42598 04900	1·11135 13427	·70127 18666	·89980 54544^{5}	·438
·563	·15857 97298	·39395 74783	1·42908 82417	1·10925 67702	·69974 68531	·90150 45270	·437
·564	·16111 85885^{+}	·39379 76292^{5}	1·43220 77080	1·10716 76608	·69822 27469	·90320 55717	·436
·565	·16365 84862^{5}	·39363 52408	1·43533 89672^{5}	1·10508 39836	·69669 95412	·90490 85995	·435
·566	·16619 94402	·39347 03119	1·43848 20985^{+}	1·10300 57080^{5}	·69517 72295^{+}	·90661 36219	·434
·567	·16874 14676	·39330 28416	1·44163 71815^{5}	1·10093 28035	·69365 58052	·90832 06502^{5}	·433
·568	·17128 45859	·39313 28286	1·44480 42968	1·09886 52398	·69213 52617	·91002 96959	·432
569	·17382 88125^{+}	·39296 02720	1·44798 35253	1·09680 29867	·69061 55923	·91174 07703^{5}	·431
·570	·17637 41648	·39278 51706	1·45117 49491	1·09474 60143	·68909 67906	·91345 38852	·430
·571	·17892 06603	·39260 75233	1·45437 86507	1·09269 42927	·68757 88499	·91516 90520	·429
·572	·18146 83166	·39242 73289	1·45759 47133	1·09064 77921	·68606 17638	·91688 62825^{+}	·428
·573	·18401 71512	·39224 45863	1·46082 32211	1·08860 64842	·68454 55258	·91860 55885^{-}	·427
·574	·18656 71819	·39205 92942	1·46406 42588	1·08657 03384	·68303 01293	·92032 69817	·426
·575	·18911 84263	·39187 14515	1·46731 79120	1·08453 93263	·68151 55678^{5}	·92205 04741^{5}	·425
·576	·19167 09023	·39168 10569^{5}	1·47058 42669	1·08251 34187	·68000 18350^{-}	·92377 60777	·424
·577	·19422 46276	·39148 81093	1·47386 34107	1·08049 25870	·67848 89242^{5}	·92550 38045^{-}	·423
·578	·19677 96203	·39129 26073	1·47715 54311^{5}	1·07847 68027	·67697 68292	·92723 36665^{+}	·422
·579	·19933 58981	·39109 45496	1·48046 04169^{5}	1·07646 60372	·67546 55434	·92896 56761	·421
·580	·20189 34792	·39089 39350	1·48377 84575^{5}	1·07446 02624	·67395 50604	·93069 98453	·420
·581	·20445 23816	·39069 07622	1·48710 96432	1·07245 94501	·67244 53739	·93243 61867	·419
·582	·20701 26234	·39048 50298	1·49045 40649	1·07046 35724	·67093 64774	·93417 47125	·418
·583	·20957 42230	·39027 67365	1·49381 18147	1·06847 26016	·66942 83645^{+}	·93591 54353	·417
·584	·21213 71984	·39006 58809	1·49718 29851	1·06648 65099^{5}	·66792 10290	·93765 83676	·416
·585	·21470 15680^{5}	·38985 24617	1·50056 76699	1·06450 52701	·66641 44644	·93940 35221	·415
·586	·21726 73504	·38963 64773	1·50396 59634	1·06252 88547	·66490 86644	·94115 09114	·414
·587	·21983 45638^{5}	·38941 79265	1·50737 79610	1·06055 72366	·66340 36226^{5}	·94290 05484	·413
·588	·22240 32270	·38919 68077	1·51080 37588	1·05859 03888	·66189 93328	·94465 24459	·412
·589	·22497 33584	·38897 31195	1·51424 34539	1·05662 82845^{-}	·66039 57887	·94640 66169	·411
·590	·22754 49767	·38874 68605^{-}	1·51769 71442	1·05467 08968	·65889 29839	·94816 30744	·410
·591	·23011 81007	·38851 80291	1·52116 49287	1·05271 81994	·65739 09122	·94992 18315	·409
·592	·23269 27492	·38828 66238	1·52464 69070	1·05077 01657	·65588 95672	·95168 29014	·408
·593	·23526 89411^{5}	·38805 26431	1·52814 31800	1·04882 67694	·65438 89428	·95344 62975^{-}	·407
·594	·23784 66954	·38781 60854	1·53165 38493	1·04688 79845^{+}	·65288 90326	·95521 20330	·406
·595	·24042 60312	·38757 69492	1·53517 90175^{-}	1·04495 37850^{5}	·65138 98305^{+}	·95698 01214	·405
·596	·24300 69674	·38733 52328	1·53871 87881	1·04302 41450^{-}	·64989 13302	·95875 05762	·404
·597	·24558 95234	·38709 09347	1·54227 32658	1·04109 90387	·64839 35254	·96052 34111	·403
·598	·24817 37185^{-}	·38684 40532	1·54584 25561	1·03917 84407	·64689 64100	·96229 86398	·402
·599	·25075 95719	·38659 45867	1·54942 67655^{-}	1·03726 23254^{5}	·64539 99778	·96407 62760	·401
·600	·25334 71031	·38634 25335^{-}	1·55302 60015^{-}	1·03535 06677	·64390 42225^{-}	·96585 63337	·400
·601	·25593 63317^{5}	·38608 78919	1·55664 03728	1·03344 34421^{5}	·64240 91380	·96763 88269	·399
·602	·25852 72773	·38583 06603	1·56026 99889	1·03154 06239	·64091 47180	·96942 37695^{-}	·398
·603	·26111 99595^{5}	·38557 08368	1·56391 49605^{+}	1·02964 21879	·63942 09565	·97121 11758	·397
·604	·26371 43982	·38530 84198	1·56757 53994	1·02774 81096	·63792 78473	·97300 10600	·396
·605	·26631 06132	·38504 34074	1·57125 14183^{5}	1·02585 83640	·63643 53841	·97479 34364^{5}	·395
·606	·26890 86244	·38477 57979	1·57494 31312	1·02397 29269	·63494 35609^{5}	·97658 83196	·394
·607	·27150 84520	·38450 55895^{+}	1·57865 06529^{5}	1·02209 17737	·63345 23716	·97838 57240	·393
·608	·27411 01160^{5}	·38423 27804	1·58237 40997^{5}	1·02021 48801	·63196 18099	·98018 56643	·392
·609	·27671 36367^{5}	·38395 73687	1·58611 35888	1·01834 22220	·63047 18698	·98198 81552	·391
·610	·27931 90345^{-}	·38367 93525^{+}	1·58986 92385^{-}	1·01647 37754	·62898 25451	·98379 32116	·390
·611	·28192 63296	·38339 87300	1·59364 11683	1·01460 95163	·62749 38298	·98560 08483^{5}	·389
·612	·28453 55427	·38311 54992	1·59742 94991	1·01274 94210	·62600 57177	·98741 10804^{5}	·388
·613	·28714 66943	·38282 96583	1·60123 43526	1·01089 34657	·62451 82027	·98922 39231^{5}	·387
·614	·28975 98052^{5}	·38254 12052	1·60505 58520	1·00904 16268	·62303 12788	·99103 93916	·386
·615	·29237 48962^{5}	·38225 01380	1·60889 41216	1·00719 38810	·62154 49398	·99285 75013	·385
·616	·29499 19882	·38195 64547	1·61274 92870	1·00535 02049	·62005 91797	·99467 82675	·384
·617	·29761 11022^{5}	·38166 01533	1·61662 14749	1·00351 05752	·61857 39924	·99650 17058	·383
·618	·30023 22594	·38136 12318	1·62051 08135	1·00167 49688	·61708 93719	·99832 78320	·382
·619	·30285 54809	·38105 96881	1·62441 74319	0·99984 33628	·61560 53120	1·00015 66618	·381
·620	·30548 07881	·38075 55202	1·62834 14610	0·99801 57342	·61412 18067	1·00198 82110	·380

$\frac{1}{2}(1+\alpha_x)$	x	z	$\frac{\frac{1}{2}(1+\alpha_x)}{z}$	$\frac{\frac{1}{2}(1-\alpha_x)}{z}$	$\frac{z}{\frac{1}{2}(1+\alpha_x)}$	$\frac{z}{\frac{1}{2}(1-\alpha_x)}$	$\frac{1}{2}(1-\alpha_x)$
·620	·30548 07881	·38075 55202	1·62834 14610	0·99801 57342	·61412 18067	1·00198 82110	·380
·621	·30810 82025^{-}	·38044 87258^{5}	1·63228 30326^{5}	·99619 20602	·61263 88500	1·00382 24956^{5}	·379
·622	·31073 77455	·38013 93031	1·63624 22801	·99437 23182	·61115 64358	1·00565 95319	·378
·623	·31336 94389	·37982 72496	1·64021 93381	·99255 64855^{-}	·60967 45580	1·00749 93359	·377
·624	·31600 33044	·37951 25634^{5}	1·64421 43425	·99074 45397	·60819 32107	1·00934 19241	·376
·625	·31863 93640	·37919 52423	1·64822 74308	·98893 64585^{-}	·60671 23877	1·01118 73128	·375
·626	·32127 76396	·37887 52840	1·65225 87417	·98713 22195	·60523 20830^{5}	1·01303 55187	·374
·627	·32391 81533	·37855 26863	1·65630 84155	·98533 18006	·60375 22907	1·01488 65584	·373
·628	·32656 09274	·37822 74469	1·66037 65938	·98353 51798	·60227 30046	1·01674 04487	·372
·629	·32920 59843	·37789 95637	1·66446 34196	·98174 23349	·60079 42189	1·01859 72066^{5}	·371
·630	·33185 33464	·37756 90342	1·66856 90374	·97995 32443	·59931 59273	1·02045 68492	·370
·631	·33450 30364	·37723 58562	1·67269 35937	·97816 78860	·59783 81239	1·02231 93935	·369
·632	·33715 50769^{5}	·37690 00273	1·67683 72357	·97638 62385^{+}	·59636 08028	1·02418 48569	·368
·633	·33980 94910	·37656 15453	1·68100 01126^{5}	·97460 82802	·59488 39578	1·02605 32568	·367
·634	·34246 63014	·37622 04076^{-}	1·68518 23752	·97283 39894	·59340 75829	1·02792 46108	·366
·635	·34512 55314	·37587 66118^{5}	1·68938 41755	·97106 33450^{-}	·59193 16722	1·02979 89366	·365
·636	·34778 72042	·37553 01557	1·69360 56675	·96929 63254	·59045 62196	1·03167 62519	·364
·637	·35045 13432	·37518 10366	1·69784 70067	·96753 29096	·58898 12192	1·03355 65747	·363
·638	·35311 79719	·37482 92522	1·70210 83501^{5}	·96577 30764	·58750 66648	1·03543 99231	·362
·639	·35578 71140	·37447 47998	1·70638 98566	·96401 68047	·58603 25506	1·03732 63153^{5}	·361
·640	·35845 87932	·37411 76771	1·71069 16865	·96226 40737	·58455 88705^{-}	1·03921 57697	·360
·641	·36113 30335^{-}	·37375 78814	1·71501 40021	·96051 48623	·58308 56184	1·04110 83047	·359
·642	·36380 98589	·37339 54102	1·71935 69672	·95876 91499	·58161 27884	1·04300 39390	·358
·643	·36648 92938	·37303 02608	1·72372 07475^{-}	·95702 69158	·58014 03745	1·04490 26914	·357
·644	·36917 13624	·37266 24307	1·72810 55103	·95528 81392^{5}	·57866 83707	1·04680 45806	·356
·645	·37185 60893	·37229 19172	1·73251 14250	·95355 27998	·57719 67709	1·04870 96259	·355
·646	·37454 34991	·37191 87176	1·73693 86626	·95182 08770	·57572 55691	1·05061 78464	·354
·647	·37723 36166	·37154 28293	1·74138 73959	·95009 23505^{-}	·57425 47594	1·05252 92615^{-}	·353
·648	·37992 64669	·37116 42495^{-}	1·74585 77998	·94836 71999	·57278 43356	1·05444 38906	·352
·649	·38262 20750	·37078 29755^{-}	1·75035 00510	·94664 54051	·57131 42919	1·05636 17534	·351
·650	·38532 04663	·37039 90044	1·75486 43280^{5}	·94492 69459	·56984 46222	1·05828 28698	·350
·651	·38802 16661	·37001 23336	1·75940 08115^{+}	·94321 18022	·56837 53204	1·06020 72596	·349
·652	·39072 57000	·36962 29601^{5}	1·76395 96841	·94149 99541	·56690 63806	1·06213 49429^{5}	·348
·653	·39343 25939^{5}	·36923 08812	1·76854 11302	·93979 13816	·56543 77967	1·06406 59402	·347
·654	·39614 23737	·36883 60940	1·77314 53365^{6}	·93808 60649	·56396 95627	1·06600 02717	·346
·655	·39885 50655^{-}	·36843 85955^{+}	1·77777 24917	·93638 39842	·56250 16725^{5}	1·06793 79580	·345
·656	·40157 06954^{5}	·36803 83829	1·78242 27866	·93468 51198	·56103 41203	1·06987 90200	·344
·657	·40428 92901	·36763 54531	1·78709 64141	·93298 94521	·55956 68998	1·07182 34785^{5}	·343
·658	·40701 08761	·36722 98033	1·79179 35692	·93129 69615^{+}	·55810 00050^{+}	1·07377 13547	·342
·659	·40973 54801	·36682 14304	1·79651 44493	·92960 76285^{5}	·55663 34300	1·07572 26697^{5}	·341
·660	·41246 31293	·36641 03313	1·80125 92538	·92792 14338	·55516 71687	1·07767 74451	·340
·661	·41519 38506	·36599 65031	1·80602 81844	·92623 83578	·55370 12150^{-}	1·07963 57024	·339
·662	·41792 76715^{-}	·36557 99426	1·81082 14452	·92455 83814	·55223 55628	1·08159 74633	·338
·663	·42066 46195^{-}	·36516 06467	1·81563 92425^{+}	·92288 14853	·55077 02062	1·08356 27499	·337
·664	·42340 47222	·36473 86123	1·82048 17850^{+}	·92120 76502^{5}	·54930 51390	1·08553 15842^{5}	·336
·665	·42614 80077	·36431 38362	1·82534 92838	·91953 68572^{5}	·54784 03552	1·08750 39887	·335
·666	·42889 45039	·36388 63152	1·83024 19523	·91786 90872	·54637 58487	1·08947 99857	·334
·667	·43164 42392	·36345 60461	1·83516 00066	·91620 43211^{5}	·54491 16134	1·09145 95979^{5}	·333
·668	·43439 72421^{5}	·36302 30256^{5}	1·84010 36651	·91454 25401	·54344 76432	1·09344 28483	·332
·669	·43715 35413	·36258 72505^{+}	1·84507 31485^{5}	·91288 37253	·54198 39320	1·09542 97599	·331
·670	·43991 31655^{5}	·36214 87175^{-}	1·85006 86809	·91122 78578	·54052 04738	1·09742 03559	·330
·671	·44267 61441	·36170 74231	1·85509 04880	·90957 49189	·53905 72624	1·09941 46598	·329
·672	·44544 25062	·36126 33640	1·86013 87987	·90792 48898^{5}	·53759 42917	1·10141 26952	·328
·673	·44821 22813^{5}	·36081 65369	1·86521 38445^{-}	·90627 77521	·53613 15556	1·10341 44860	·327
·674	·45098 54993	·36036 69383	1·87031 58595^{+}	·90463 34869^{5}	·53466 90480	1·10542 00562	·326
·675	·45376 21901	·35991 45648	1·87544 50808	·90299 20759	·53320 67626	1·10742 94301	·325
·676	·45654 23838	·35945 94128	1·88060 17480	·90135 35005^{+}	·53174 46934^{5}	1·10944 26320	·324
·677	·45932 61108	·35900 14788	1·88578 61038	·89971 77423	·53028 28343	1·11145 96868	·323
·678	·46211 34017	·35854 07594	1·89099 83936	·89808 47828	·52882 11790	1·11348 06191	·322
·679	·46490 42874^{5}	·35807 72508	1·89623 88659	·89645 46038	·52735 97214	1·11550 54543	·321
·680	·46769 87991	·35761 09496	1·90150 77721	·89482 71869	·52589 84553	1·11753 42174	·320

$\frac{1}{2}(1+\alpha_x)$	x	z	$\frac{\frac{1}{2}(1+\alpha_x)}{z}$	$\frac{\frac{1}{2}(1-\alpha_x)}{z}$	$\frac{z}{\frac{1}{2}(1+\alpha_x)}$	$\frac{z}{\frac{1}{2}(1-\alpha_x)}$	$\frac{1}{2}(1-\alpha_x)$
·680	·46769 87991	·35761 09496	1·90150 77721	·89482 71869	·52589 84553	1·11753 42174	·320
·681	·47049 69679	·35714 18520	1·90680 53665^{+}	·89320 25138	·52443 73744^{5}	1·11956 69342	·319
·682	·47329 88254	·35666 99544	1·91213 19067	·89158 05665^{-}	·52297 64727	1·12160 36302	·318
·683	·47610 44034^{5}	·35619 52531	1·91748 76533	·88996 13266	·52151 57439	1 12364 43316	·317
·684	·47891 37341	·35571 77443^{5}	1·92287 28700	·88834 47762	·52005 51818	1·12568 90644	·316
·685	·48172 68495^{5}	·35523 74244	1·92828 78239	·88673 08971	·51859 47801	1·12773 78552	·315
·686	·48454 37824	·35475 42894	1·93373 27850^{+}	·88511 96713	·51713 45326	1·12979 07305^{6}	·314
·687	·48736 45654^{5}	·35426 83355^{+}	1·93920 80271	·88351 10808	·51567 44331	1·13184 77173	·313
·688	·49018 92317	·35377 95589^{5}	1·94471 38270	·88190 51076	·51421 44752	1·13390 88428	·312
·689	·49301 78145	·35328 79558	1·95025 04650^{+}	·88030 17338^{5}	·51275 46528	1·13597 41343	·311
·690	·49585 03473	·35279 35220	1·95581 82250^{-}	·87870 09417	·51129 49594	1·13804 36194	·310
·691	·49868 68641	·35229 62538	1·96141 73938	·87710 27130	·50983 53890	1·14011 73262^{5}	·309
·692	·50152 73993	·35179 61469	1·96704 82640	·87550 70308	·50837 59348	1·14219 52821	·308
·693	·50437 19864	·35129 31975^{5}	1·97271 11280	·87391 38763	·50691 65910	1·14427 75164^{5}	·307
·694	·50722 06606	·35078 74016	1·97840 62849	·87232 32323	·50545 73510	1·14636 40575^{+}	·306
·695	·51007 34570	·35027 87549	1·98413 40370	·87073 50810	·50399 82085^{-}	1·14845 49341	·305
·696	·51293 04106	·34976 72533	1·98989 46898	·86914 94047	·50253 91570	1·15055 01753	·304
·697	·51579 15570	·34925 28927	1·99568 85530	·86756 61859	·50108 01903	1·15264 98108	·303
·698	·51865 69321	·34873 56688	2·00151 59402^{5}	·86598 54068	·49962 13020	1·15475 38701	·302
·699	·52152 65718	·34821 55774	2·00737 71692	·86440 70500^{-}	·49816 24855^{+}	1·15686 23833	·301
·700	·52440 05127	·34769 26142	2·01327 25614^{5}	·86283 10978	·49670 37346	1·15897 53807	·300
·701	·52727 87914	·34716 67749	2·01920 24429	·86125 75327	·49524 50427	1·16109 28927	·299
·702	·53016 14450^{+}	·34663 80552	2·02516 71435^{-}	·85968 63373	·49378 64034	1·16321 49502	·298
·703	·53304 85109	·34610 64506	2·03116 69975^{+}	·85811 74940	·49232 78102	1·16534 15844	·297
·704	·53594 00266	·34557 19567	2·03720 23436	·85655 09854	·49086 92566	1·16747 28265^{+}	·296
·705	·53883 60303	·34503 45690	2·04327 35248	·85498 67941	·48941 07362	1·16960 87085^{-}	·295
·706	·54173 65601	·34449 42831	2·04938 08886	·85342 49026	·48795 22423	1·17174 92622	·294
·707	·54464 16548	·34395 10944	2·05552 47871	·85186 52936^{5}	·48649 37685^{+}	1·17389 45200	·293
·708	·54755 13533	·34340 49982	2·06170 55769	·85030 79498	·48503 53082^{5}	1·17604 45145^{+}	·292
·709	·55046 56950^{+}	·34285 59901	2·06792 36194	·84875 28537	·48357 68549	1·17819 92787	·291
·710	·55338 47196	·34230 40653	2·07417 92809	·84719 99880	·48211 84018	1·18035 88458	·290
·711	·55630 84670	·34174 92191	2·08047 29324^{5}	·84564 93354	·48065 99425^{-}	1·18252 32494	·289
·712	·55923 69776	·34119 14468	2·08680 49500	·84410 08786	·47920 14702	1·18469 25235^{6}	·288
·713	·56217 02922^{5}	·34063 07435^{+}	2·09317 57146	·84255 46004	·47774 29783	1·18686 67022	·287
·714	·56510 84520	·34006 71046	2·09958 56124	·84101 04834	·47628 44602	1·18904 58202	·286
·715	·56805 14983	·33950 05250^{6}	2·10603 50348^{5}	·83946 85104	·47482 59091	1·19122 99123	·285
·716	·57099 94731	·33893 09999	2·11252 43786	·83792 86641	·47336 73183	1·19341 90138	·284
·717	·57395 24186	·33835 85244	2·11905 40456	·83639 09273^{5}	·47190 86812	1·19561 31604	·283
·718	·57691 03773	·33778 30934	2·12562 44436	·83485 52829	·47044 99908	1·19781 23880	·282
·719	·57987 33924	·33720 47020	2·13223 59855^{+}	·83332 17134	·46899 12406	1·20001 67329	·281
·720	·58284 15073	·33662 33449	2·13888 90902^{5}	·83179 02018	·46753 24235^{+}	1·20222 62319	·280
·721	·58581 47657	·33603 90172	2·14558 41823	·83026 07307	·46607 35329	1·20444 09220	·279
·722	·58879 32119	·33545 17137	2·15232 16921	·82873 32831	·46461 45619	1·20666 08406	·278
·723	·59177 68906	·33486 14291	2·15910 20561	·82720 78417	·46315 55035^{+}	1·20888 60255^{-}	·277
·724	·59476 58468	·33426 81581	2·16592 57168	·82568 43893	·46169 63510	1·21111 65150	·276
·725	·59776 01260	·33367 18956	2·17279 31227	·82416 29086	·46023 70974	1·21335 23476	·275
·726	·60075 97742^{5}	·33307 26361	2·17970 47290	·82264 33826	·45877 77357	1·21559 35624	·274
·727	·60376 48378	·33247 03742	2·18666 09970	·82112 57939	·45731 82589	1·21784 01987	·273
·728	·60677 53635^{+}	·33186 51046	2·19366 23945^{-}	·81961 01254	·45585 86602	1·22009 22963	·272
·729	·60979 13987	·33125 68217	2·20070 93961	·81809 63599	·45439 89323	1·22234 98955^{-}	·271
·730	·61281 29910	·33064 55199^{5}	2·20780 24832	·81658 44801	·45293 90684	1·22461 30368	·270
·731	·61584 01887	·33003 11938	2·21494 21439	·81507 44688	·45147 90613	1·22688 17614	·269
·732	·61887 30405^{+}	·32941 38377	2·22212 88733	·81356 63088	·45001 89039	1·22915 61108	·268
·733	·62191 15956^{-}	·32879 34458^{5}	2·22936 31739	·81205 99829	·44855 85891^{5}	1·23143 61268	·267
·734	·62495 59035^{-}	·32817 00125^{+}	2·23664 55558	·81055 54739	·44709 81097	1·23372 18515^{+}	·266
·735	·62800 60144	·32754 35321	2·24397 65343	·80905 27641	·44563 74586^{5}	1·23601 33287	·265
·736	·63106 19790^{5}	·32691 39986	2·25135 66356	·80755 18367	·44417 66285^{+}	1·23831 06075^{5}	·264
·737	·63412 38485^{+}	·32628 14062	2·25878 63913	·80605 26742	·44271 56122	1·24061 37117	·263
·738	·63719 16745^{6}	·32564 57489	2·26626 63414^{5}	·80455 52594	·44125 44023	1·24292 27058	·262
·739	·64026 55092	·32500 70208	2·27379 70340	·80305 95749^{5}	·43979 29916^{5}	1·24523 76277	·261
·740	·64334 54054	·32436 52159	2·28137 90252	·80156 56034^{5}	·43833 13728	1·24755 85226	·260

$\frac{1}{2}(1+\alpha_x)$	x	z	$\frac{\frac{1}{2}(1+\alpha_x)}{z}$	$\frac{\frac{1}{2}(1-\alpha_x)}{z}$	$\frac{z}{\frac{1}{2}(1+\alpha_x)}$	$\frac{z}{\frac{1}{2}(1-\alpha_x)}$	$\frac{1}{2}(1-\alpha_x)$
·740	·64334 54054	·32436 52159	2·28137 90252	·80156 56034^{5}	·43833 13728	1·24755 85226	·260
·741	·64643 14163	·32372 03280	2·28901 28792	·80007 33275^{5}	·43686 95384	1·24988 54362	·259
·742	·64952 35958	·32307 23510	2·29669 91689	·79858 27299	·43540 74811	1·25221 84147	·258
·743	·65262 19983^{5}	·32242 12787	2·30443 84756	·79709 37930	·43394 51934	1·25455 75047	·257
·744	·65572 66788	·32176 71049	2·31223 13893^{5}	·79560 64996	·43248 26679	1·25690 27535^{6}	·256
·745	·65883 76927	·32110 98232	2·32007 85092	·79412 08320	·43101 98970	1·25925 42088	·255
·746	·66195 50963	·32044 94274	2·32798 04433	·79263 67729	·42955 68732	1·26161 19188	·254
·747	·66507 89462	·31978 59109	2·33593 78089	·79115 43047	·42809 35889	1·26397 59324	·253
·748	·66820 92997	·31911 92673	2·34395 12327	·78967 34099	·42663 00365^{+}	1·26634 62989	·252
·749	·67134 62149	·31844 94901	2·35202 13511^{5}	·78819 40709	·42516 62084	1·26872 30682	·251
·750	·67448 97502	·31777 65727	2·36014 88104	·78671 62701	·42370 20969	1·27110 62907	·250
·751	·67763 99649	·31710 05084	2·36833 42667	·78523 99899	·42223 76942^{5}	1·27349 60176	·249
·752	·68079 69188	·31642 12905^{+}	2·37657 83864	·78376 52125^{+}	·42077 29927	1·27589 23004	·248
·753	·68396 06724	·31573 89123	2·38488 18461	·78229 19203	·41930 79844	1·27829 51914	·247
·754	·68713 12868	·31505 33669	2·39324 53332	·78082 00954	·41784 26616^{5}	1·28070 47434	·246
·755	·69030 88240	·31436 46474	2·40166 95461	·77934 97202	·41637 70164	1·28312 10098	·245
·756	·69349 33463	·31367 27469	2·41015 51936	·77788 07768	·41491 10409	1·28554 40446	·244
·757	·69668 49171	·31297 76583^{5}	2·41870 29962	·77641 32471	·41344 47270	1·28797 39027	·243
·758	·69988 36002	·31227 93747	2·42731 36859	·77494 71134	·41197 80669	1·29041 06392	·242
·759	·70308 94604	·31157 78888	2·43598 80062	·77348 23577	·41051 10524	1·29285 43102	·241
·760	·70630 25629	·31087 31933^{5}	2·44472 67125^{+}	·77201 89619	·40904 36755^{-}	1·29530 49723	·240
·761	·70952 29739	·31016 52812	2·45353 05727	·77055 69078^{5}	·40757 59280	1·29776 26828	·239
·762	·71275 07602^{5}	·30945 41449^{5}	2·46240 03667	·76909 61775^{+}	·40610 78018	1·30022 74998	·238
·763	·71598 59896	·30873 97772	2·47133 68874	·76763 67527	·40463 92886	1·30269 94818	·237
·764	·71922 87305^{-}	·30802 21705^{-}	2·48034 09405^{+}	·76617 86151	·40317 03802	1·30517 86884	·236
·765	·72247 90519	·30730 13172	2·48941 33450^{-}	·76472 17465	·40170 10682^{5}	1·30766 51796	·235
·766	·72573 70241	·30657 72098	2·49855 49332	·76326 61284	·40023 13444	1·31015 90163	·234
·767	·72900 27178	·30584 98406	2·50776 65515^{-}	·76181 17425^{-}	·39876 12002^{5}	1·31266 02600	·233
·768	·73227 62048	·30511 92018	2·51704 90599^{5}	·76035 85702	·39729 06273	1·31516 89732	·232
·769	·73555 75574	·30438 52859	2·52640 33306	·75890 65921	·39581 96175^{+}	1·31768 52202	·231
·770	·73884 68492	·30364 80841	2·53583 02605^{6}	·75745 57921	·39434 81611	1·32020 90609^{5}	·230
·771	·74214 41544	·30290 75892	2·54533 07461^{5}	·75600 61490	·39287 62506	1·32274 05641	·229
·772	·74544 95482	·30216 37930	2·55490 57096	·75455 76448	·39140 38770	1·32527 97939	·228
·773	·74876 31066	·30141 66874	2·56455 60860	·75311 02607	·38993 10315^{+}	1·32782 68166	·227
·774	·75208 49067	·30066 62640^{5}	2·57428 28263	·75166 39777	·38845 77055^{-}	1·33038 16993	·226
·775	·75541 50264	·29991 25148	2·58408 68980	·75021 87769^{5}	·38698 38900	1·33294 45101	·225
·776	·75875 35445^{+}	·29915 54312	2·59396 92891	·74877 46390	·38550 95763	1·33551 53178^{5}	·224
·777	·76210 05410	·29839 50049	2·60393 09886^{5}	·74733 15450^{-}	·38403 47553	1·33809 41922	·223
·778	·76545 60967	·29763 12273	2·61397 30268	·74588 94755^{+}	·38255 94181	1·34068 12039	·222
·779	·76882 02935^{-}	·29686 40898	2·62409 64360	·74444 84112	·38108 35556	1·34327 64244	·221
·780	·77219 32142	·29609 35838	2·63430 22705^{-}	·74300 83327	·37960 71587	1·34587 99262	·220
·781	·77557 49428	·29531 97004	2·64459 16031^{5}	·74156 92203^{5}	·37813 02182	1·34849 17828	·219
·782	·77896 55644	·29454 24309	2·65496 55259	·74013 10545^{+}	·37665 27250^{-}	1·35111 20684	·218
·783	·78236 51649	·29376 17663	2·66542 51499	·73869 38155^{-}	·37517 46696	1·35374 08586	·217
·784	·78577 38315^{+}	·29297 76976	2·67597 16064	·73725 74834	·37369 60428	1·35637 82295^{6}	·216
·785	·78919 16527	·29219 02156	2·68660 60467	·73582 20383	·37221 68352	1·35902 42586	·215
·786	79261 87177	·29139 93112	2·69732 96430	·73438 74600	·37073 70371	1·36167 90242	·214
·787	·79605 51173	·29060 49750^{5}	2·70814 35886	·73295 37286	·36925 66392	1·36434 26059	·213
·788	·79950 09431	·28980 71978	2·71904 90987	·73152 08235^{6}	·36777 56317	1·36701 50840	·212
·789	·80295 62883	·28900 59700	2·73004 74105^{+}	·73008 87245^{-}	·36629 40051	1·36969 65402^{5}	·211
·790	·80642 12470	·28820 12820	2·74113 97841	·72865 74109^{5}	·36481 17494	1·37238 70573	·210
·791	·80989 59177	·28739 31243	2·75232 75027	·72722 68623	·36332 88549^{5}	1·37508 67190	·209
·792	·81338 03882	·28658 14869	2·76361 18733^{5}	·72579 70576^{5}	·36184 53118	1·37779 56103	·208
·793	·81687 47655^{-}	·28576 63602	2·77499 42277	·72436 79762	·36036 11099^{5}	1·38051 38173	·207
·794	·82037 91459^{5}	·28494 77341	2·78647 59220	·72293 95969	·35887 62394	1·38324 14275^{+}	·206
·795	·82389 36303	·28412 55985^{+}	2·79805 83380	·72151 18985^{-}	·35739 06900	1·38597 85294	·205
·796	·82741 83207	·28329 99434	2·80974 28839	·72008 48597	·35590 44515^{+}	1·38872 52128	·204
·797	·83095 33205^{5}	·28247 07584^{5}	2·82153 90045^{6}	·71865 84591	·35441 75137	1·39148 15687	·203
·798	·83449 87348	·28163 80333	2·83342 41319	·71723 26750^{+}	·35292 98663	1·39424 76896	·202
·799	·83805 46698^{5}	·28080 17575^{-}	2·84542 37865^{-}	·71580 74857	·35144 14987	1·39702 36690	·201
·800	·84162 12335^{5}	·27996 19204	2·85753 14772	·71438 28693	·34995 24005^{+}	1·39980 96021	·200

$\frac{1}{2}(1+a_x)$	x	z	$\frac{\frac{1}{2}(1+a_x)}{z}$	$\frac{\frac{1}{2}(1-a_x)}{z}$	$\frac{z}{\frac{1}{2}(1+a_x)}$	$\frac{z}{\frac{1}{2}(1-a_x)}$	$\frac{1}{2}(1-a_x)$
·800	·84162 12335^{5}	·27996 19204	2·85753 14772	·71438 28693	·34995 24005^{+}	1·39980 96021	·200
·801	·84519 85353	·27911 85114	2·86974 87526	·71295 88037	·34846 25611	1·40260 55856^{5}	·199
·802	·84878 66859	·27827 15197	2·88207 71913	·71153 52667	·34697 19697	1·40541 17158	·198
·803	·85238 57979	·27742 09344	2·89451 84029	·71011 22358	·34548 06157	1·40822 80934	·197
·804	·85599 59855^{-}	·27656 67444	2·90707 40288	·70868 96886	·34398 84881	1·41105 48186	·196
·805	·85961 73642	·27570 89387	2·91974 57427	·70726 76023	·34249 55760	1·41389 19933^{5}	·195
·806	·86325 00516	·27484 75059	2·93253 52517	·70584 59539	·34100 18684	1·41673 97213	·194
·807	·86689 41666	·27398 24348	2·94544 42969	·70442 47203	·33950 73541	1·41959 81077	·193
·808	·87054 98302	·27311 37138	2·95847 46547	·70300 38783^{5}	·33801 20220	1·42246 72591^{5}	·192
·809	·87421 71648^{5}	·27224 13312	2·97162 81372	·70158 34044^{5}	·33651 58606	1·42534 72840	·191
·810	·87789 62950	·27136 52755^{-}	2·98490 65933	·70016 32750	·33501 88586	1·42823 82927	·190
·811	·88158 73470	·27048 55347	2·99831 19097	·69874 34660	·33352 10045^{+}	1·43114 03951	·189
·812	·88529 04488	·26960 20968	3·01184 60119	·69732 39535^{e}	·33202 22867	1·43405 37063	·188
·813	·88900 57306	·26871 49497	3·02551 08652	·69590 47131^{5}	·33052 26934	1·43697 83407	·187
·814	·89273 33243	·26782 40812	3·03930 84755^{5}	·69448 57205^{-}	·32902 22128	1·43991 44151	·186
·815	·89647 33640	·26692 94789	3·05324 08909	·69306 69507	·32752 08330	1·44286 20482	·185
·816	·90022 59857	·26603 11303	3·06731 02020	·69164 83789	·32601 85420	1·44582 13603	·184
·817	·90399 13276	·26512 90227	3·08151 85439^{5}	·69022 99799	·32451 53277	1·44879 24738	·183
·818	·90776 95299	·26422 31433^{5}	3·09586 80970^{5}	·68881 17282	·32301 11777	1·45177 55129	·182
·819	·91156 07351	·26331 34793	3·11036 10881	·68739 35982	·32150 60797	1·45477 06039	·181
·820	·91536 50879	·26240 00175^{-}	3·12499 97917	·68597 55640	·32000 00213	1·45777 78750^{-}	·180
·821	·91918 27352	·26148 27447	3·13978 65314^{5}	·68455 75994	·31849 29899	1·46079 74564	·179
·822	·92301 38263	·26056 16475^{+}	3·15472 36815^{+}	·68313 96780	·31698 49727	1·46382 94806	·178
·823	·92685 85128	·25963 67125^{+}	3·16981 36678	·68172 17730	·31547 59569	1·46687 40821	·177
·824	·93071 69489	·25870 79259	3·18505 89695^{+}	·68030 38576	·31396 59295^{+}	1·46993 13974	·176
·825	·93458 92911	·25777 52740	3·20046 21205^{+}	·67888 59043^{5}	·31245 48776	1·47300 15656	·175
·826	·93847 56984	·25683 87427	3·21602 57109^{5}	·67746 78858	·31094 27877	1·47608 47280	·174
·827	·94237 63326	·25589 83178^{5}	3·23175 23888	·67604 97742	·30942 96467	1·47918 10280	·173
·828	·94629 13579^{5}	·25495 39852	3·24764 48616	·67463 15413	·30791 54411	1·48229 06117	·172
·829	·95022 09415^{+}	·25400 57303	3·26370 58979	·67321 31587	·30640 01571^{5}	1·48541 36274	·171
·830	·95416 52531	·25305 35384	3·27993 83292^{5}	·67179 45976	·30488 37812	1·48855 02260	·170
·831	·95812 44654	·25209 73948	3·29634 50520	·67037 58289	·30336 62994	1·49170 05610	·169
·832	·96209 87539	·25113 72844^{5}	3·31292 90293	·66895 68232	·30184 76977	1·49486 47884	·168
·833	·96608 82971	·25017 31922	3·32969 32922	·66753 75508	·30032 79618	1·49804 30671^{5}	·167
·834	·97009 32766	·24920 51027	3·34664 09431	·66611 79815^{-}	·29880 70776	1·50123 55585^{5}	·166
·835	·97411 38770	·24823 30005^{-}	3·36377 51567	·66469 80848^{5}	·29728 50305^{-}	1·50444 24270	·165
·836	·97815 02862	·24725 68697	3·38109 91825^{-}	·66327 78300^{5}	·29576 18059	1·50766 38396	·164
·837	·98220 26953	·24627 66945^{5}	3·39861 63471	·66185 71859	·29423 73889	1·51089 99665^{5}	·163
·838	·98627 12987	·24529 24589	3·41633 00565^{e}	·66043 61207	·29271 17648	1·51415 09809	·162
·839	·99035 62942	·24430 41465^{-}	3·43424 37985^{-}	·65901 46026	·29118 49183	1·51741 70589	·161
·840	·99445 78832	·24331 17408	3·45236 11450^{-}	·65759 25990	·28965 68343	1·52069 83799	·160
·841	·99857 62706	·24231 52251	3·47068 57548^{5}	·65617 00773	·28812 74972	1·52399 51265^{+}	·159
·842	1·00271 16650^{+}	·24131 45826	3·48922 13765^{+}	·65474 70041	·28659 68914	1·52730 74847	·158
·843	1·00686 42788	·24030 97961	3·50797 18507	·65332 33459	·28506 50012	1·53063 56436	·157
·844	1·01103 43281	·23930 08482	3·52694 11133	·65189 90683	·28353 18107	1·53397 97962	·156
·845	1·01522 20332	·23828 77215^{+}	3·54613 31983	·65047 41370	·28199 73036	1·53734 01388	·155
·846	1·01942 76182	·23727 03982	3·56555 22410	·64904 85167	·28046 14636	1·54071 68714	·154
·847	1·02365 13115^{+}	·23624 88602	3·58520 24816	·64762 21720	·27892 42742	1·54411 01976	·153
·848	1·02789 33458	·23522 30895^{-}	3·60508 82671	·64619 50667	·27738 57187	1·54752 03255^{-}	·152
·849	1·03215 39579	·23419 30674	3·62521 40578	·64476 71646	·27584 57802	1·55094 74659	·151
·850	1·03643 33895^{-}	·23315 87753	3·64558 44265^{+}	·64333 84282	·27430 44415^{-}	1·55439 18351	·150
·851	1·04073 18864	·23212 01943	3·66620 40662	·64190 88200	·27276 16854	1·55785 36527	·149
·852	1·04504 96998	·23107 73050^{-}	3·68707 77942	·64047 83023	·27121 74941	1·56133 31419	·148
·853	1·04938 70848	·23003 00883	3·70821 05495^{+}	·63904 68356	·26967 18502^{5}	1·56483 05324	·147
·854	1·05374 43022	·22897 85242	3·72960 74067	·63761 43810	·26812 47356^{5}	1·56834 60565^{-}	·146
·855	1·05812 16178	·22792 25929^{5}	3·75127 35746	·63618 08986	·26657 61321	1·57187 99514	·145
·856	1·06251 93023	·22686 22742	3·77321 44003	·63474 63477	·26502 60213	1·57543 24599	·144
·857	1·06693 76321	·22579 75475	3·79543 53775^{+}	·63331 06873	·26347 43845^{-}	1·57900 38288	·143
·858	1·07137 68892	·22472 83920	3·81794 21493	·63187 38755^{+}	·26192 12028	1·58259 43099	·142
·859	1·07583 73609	·22365 47867	3·84074 05129	·63043 58699	·26036 64571	1·58620 41608	·141
·860	1·08031 93408	·22257 67101	3·86383 64255^{-}	·62899 66274	·25881 01280	1·58983 36437	·140

$\frac{1}{2}(1+\alpha_x)$	x	z	$\frac{\frac{1}{2}(1+\alpha_x)}{z}$	$\frac{\frac{1}{2}(1-\alpha_x)}{z}$	$\frac{z}{\frac{1}{2}(1+\alpha_x)}$	$\frac{z}{\frac{1}{2}(1-\alpha_x)}$	$\frac{1}{2}(1-\alpha_x)$
·860	1·08031 93408	·22257 67101	3·86383 64255^{-}	·62899 66274	·25881 01280	1·58983 36437	·140
·861	1·08482 31279	·22149 41407	3·88723 60109	·62755 61040	·25725 21960	1·59348 30267^{5}	·139
·862	1·08934 90279	·22040 70565^{-}	3·91094 55647	·62611 42551	·25569 26409	1·59715 25832	·138
·863	1·09389 73525^{-}	·21931 54352	3·93497 15591	·62467 10355^{-}	·25413 14429	1·60084 25927	·137
·864	1·09846 84202	·21821 92542	3·95932 06524	·62322 63990	·25256 85813	1·60455 33399	·136
·865	1·10306 25561	·21711 84907	3·98399 96923	·62178 02988	·25100 40354^{5}	1·60828 51160	·135
·866	1·10768 00920	·21601 31213	4·00901 57241	·62033 26871	·24943 77844	1·61203 82188	·134
·867	1·11232 13671	·21490 31226	4·03437 59995^{+}	·61888 35155	·24786 98069	1·61581 29518	·133
·868	1·11698 67277	·21378 84706	4·06008 79819	·61743 27346	·24630 00813	1·61960 96255^{-}	·132
·869	1·12167 65277	·21266 91410	4·08615 93548	·61598 02940	·24472 85857	1·62342 85573	·131
·870	1·12639 11289	·21154 51092	4·11259 80324	·61452 61428	·24315 52980	1·62727 00710	·130
·871	1·13113 09007	·21041 63503	4·13941 21638	·61307 02286	·24158 01955^{5}	1·63113 44986	·129
·872	1·13589 62211	·20928 28289	4·16661 01464	·61161 24985^{5}	·24000 32556	1·63502 21789	·128
·873	1·14068 74762	·20814 45492^{5}	4·19420 06320	·61015 28984	·23842 44550	1·63893 34586	·127
·874	1·14550 50613	·20700 14552	4·22219 25411	·60869 13732	·23684 37702	1·64286 86918	·126
·875	1·15034 93802	·20585 35302	4·25059 50660^{5}	·60722 78667	·23526 11774	1·64682 82416	·125
·876	1·15522 08464	·20470 07474	4·27941 76924	·60576 23218	·23367 66523	1·65081 24789	·124
·877	1·16011 98829^{5}	·20354 30793	4·30867 01991	·60429 46801	·23209 01702	1·65482 17829	·123
·878	1·16504 69221	·20238 04983	4·33836 26756	·60282 48820	·23050 17065^{-}	1·65885 65433	·122
·879	1·17000 24074	·20121 29760	4·36850 55383^{5}	·60135 28670^{5}	·22891 12355^{-}	1·66291 71569	·121
·880	1·17498 67920	·20004 04838	4·39910 95366^{5}	·59987 85732	·22731 87316	1·66700 40317	·120
·881	1·18000 05403	·19886 29927	4·43018 57685^{+}	·59840 19370	·22572 41687	1·67111 75854	·119
·882	1·18504 41279	·19768 04728	4·46174 57036	·59692 28946	·22412 75201	1·67525 82437	·118
·883	1·19011 80420	·19649 28942	4·49380 11811	·59544 13796	·22252 87590	1·67942 64461	·117
·884	1·19522 27816	·19530 02264	4·52636 44411^{5}	·59395 73249	·22092 78579	1·68362 26411	·116
·885	1·20035 88581	·19410 24382	4·55944 81355^{+}	·59247 06617	·21932 47889	1·68784 72888	·115
·886	1·20552 67961	·19289 94980	4·59306 53476	·59098 13201	·21771 95237	1·69210 08600	·114
·887	1·21072 71329	·19169 13738	4·62722 96054	·58948 92282	·21611 20336	1·69638 38390	·113
·888	1·21596 04197	·19047 80328	4·66195 49084	·58799 43128	·21450 22892	1·70069 67215^{+}	·112
·889	1·22122 72221	·18925 94418	4·69725 57433	·58649 64989	·21289 02607	1·70504 00163	·111
·890	1·22652 81200	·18803 55670	4·73314 71070	·58499 57099	·21127 59180	1·70941 42455	·110
·891	1·23186 37089	·18680 63740	4·76964 45309	·58349 18674	·20965 92301	1·71381 99448	·109
·892	1·23723 45993	·18557 18278	4·80676 41009	·58198 48911	·20804 01657	1·71825 76648	·108
·893	1·24264 14187	·18433 18928	4·84452 24882	·58047 46990	·20641 86929^{5}	1·72272 79702	·107
·894	1·24808 48112	·18308 65328	4·88293 69721	·57896 12070	·20479 47794	1·72723 14413	·106
·895	1·25356 54386	·18183 57108	4·92202 54712	·57744 43290	·20316 83919	1·73176 86740	·105
·896	1·25908 39805^{+}	·18057 93893	4·96180 65699	·57592 39769	·20153 94969	1·73634 02813	·104
·897	1·26464 11358	·17931 75299	5·00229 95540	·57440 00603	·19990 80607	1·74094 68925^{5}	·103
·898	1·27023 76225^{+}	·17805 00939	5·04352 44402	·57287 24865	·19827 40466	1·74558 91556	·102
·899	1·27587 41794	·17677 70413	5·08550 20158^{5}	·57134 11608^{5}	·19663 74208	1·75026 77359	·101
·900	1·28155 15658	·17549 83319	5·12825 38719	·56980 59858	·19499 81465^{+}	1·75498 33188	·100
·901	1·28727 05633	·17421 39243	5·17180 24462^{5}	·56826 68615^{-}	·19335 61868	1·75973 66093^{5}	·099
·902	1·29303 19763	·17292 37766	5·21617 10643^{5}	·56672 36855^{-}	·19171 15040	1·76452 83330	·098
·903	1·29883 66327	·17162 78460	5·26138 39837	·56517 63526	·19006 40598	1·76935 92369	·097
·904	1·30468 53854^{5}	·17032 60887	5·30746 64436	·56362 47551	·18841 38149	1·77423 00902	·096
·905	1·31057 91123	·16901 84602	5·35444 47090	·56206 87816	·18676 07295	1·77914 16864	·095
·906	1·31651 87185^{5}	·16770 49152	5·40234 61337	·56050 83185^{0}	·18510 47629	1·78409 48421	·094
·907	1·32250 51368	·16638 54072	5·45119 92086	·55894 32485^{+}	·18344 58734	1·78909 03999	·093
·908	1·32853 93290	·16505 98890	5·50103 36292	·55737 34514	·18178 40187	1·79412 92278	·092
·909	1·33462 22868	·16372 83123	5·55188 03522	·55579 88031	·18011 91554	1·79921 22228	·091
·910	1·34075 50338	·16239 06278	5·60377 16730^{5}	·55421 91764^{5}	·17845 12393	1·80434 03088	·090
·911	1·34693 86262	·16104 67852^{5}	5·65674 12916^{5}	·55263 44401	·17678 02253	1·80951 44410	·089
·912	1·35317 41546	·15969 67332^{5}	5·71082 43946	·55104 44591	·17510 60672	1·81473 56051^{5}	·088
·913	1·35946 27455^{-}	·15834 04193	5·76605 77398	·54944 90946	·17342 87177	1·82000 48191	·087
·914	1·36580 55627	·15697 77897	5·82247 97405^{5}	·54784 82032	·17174 81287	1·82532 31356	·086
·915	1·37220 38091	·15560 87897	5·88013 05635	·54624 16370^{5}	·17006 42510	1·83069 16430	·085
·916	1·37865 87286	·15423 33632	5·93905 22329	·54462 92441	·16837 70340	1·83611 14665^{-}	·084
·917	1·38517 16082	·15285 14529	5·99928 87383^{5}	·54301 08673	·16668 64263	1·84158 37698	·083
·918	1·39174 37794	·15146 30002	6·06088 61488	·54138 63444^{5}	·16499 23749^{5}	1·84710 97586	·082
·919	1·39837 66208	·15006 79451	6·12389 27423	·53975 55083	·16329 48260	1·85269 06805^{+}	·081
·920	1·40507 15603	·14866 62263	6·18835 91389	·53811 81860	·16159 37242	1·85832 78284	·080

$\frac{1}{2}(1+\alpha_x)$	x	z	$\frac{\frac{1}{2}(1+\alpha_x)}{z}$	$\frac{\frac{1}{2}(1-\alpha_x)}{z}$	$\frac{z}{\frac{1}{2}(1+\alpha_x)}$	$\frac{z}{\frac{1}{2}(1-\alpha_x)}$	$\frac{1}{2}(1-\alpha_x)$
·920	1·40507 15603	·14866 62263	6·18835 91389	·53811 81860	·16159 37242	1·85832 78284	·080
·921	1·41183 00774	·14725 77808	6·25433 84457	·53647 41989	·15988 90128	1·86402 25420	·079
·922	1·41865 37061	·14584 25444	6·32188 64133	·53482 33625^{+}	·15818 06339	1·86977 62104	·078
·923	1·42554 40371	·14442 04512$_{5}$	6·39106 16025^{+}	·53316 54858	·15646 85278	1·87559 02747$_{5}$	·077
·924	1·43250 27208	·14299 14335$_{5}$	6·46192 55649$_{5}$	·53150 03711	·15475 26337	1·88146 62309	·076
·925	1·43953 14708	·14155 54224	6·53454 30392	·52982 78140	·15303 28891	1·88740 56318	·075
·926	1·44663 20671	·14011 23467	6·60898 21635^{-}	·52814 76027	·15130 92297	1·89341 00901	·074
·927	1·45380 63589	·13866 21337	6·68531 46959	·52645 95176	·14958 15897	1·89948 12832	·073
·928	1·46105 62691	·13720 47087	6·76361 62673	·52476 33311	·14784 99017	1·90562 09547$_{5}$	·072
·929	1·46838 37982	·13573 99953	6·84396 66452$_{5}$	·52305 88071	·14611 40961	1·91183 09192	·071
·930	1·47579 10282	·13426 79144	6·92645 00320	·52134 57013	·14437 41015^{+}	1·91811 30629	·070
·931	1·48328 01274	·13278 83859	7·01115 53337$_{5}$	·51962 37573	·14262 98452	1·92446 93607	·069
·932	1·49085 33552	·13130 13263	7·09817 65869$_{5}$	·51789 27123$_{5}$	·14088 12514	1·93090 18570	·068
·933	1·49851 30679	·12980 66504	7·18761 32490$_{5}$	·51615 22912	·13912 82426	1·93741 26921	·067
·934	1·50626 17234	·12830 42705^{-}	7·27957 06377	·51440 22078	·13737 07393	1·94400 40980	·066
·935	1·51410 18877$_{5}$	·12679 40964	7·37416 03637$_{5}$	·51264 21643	·13560 86592$_{5}$	1·95067 84061	·065
·936	1·52203 62418	·12527 60353	7·47150 08155^{+}	·51087 18506	·13384 19181	1·95743 80517$_{5}$	·064
·937	1·53006 75881	·12374 99916	7·57171 76848	·50909 09436	·13207 04286	1·96428 55812	·063
·938	1·53819 88586	·12221 58668	7·67494 45411	·50729 91061	·13029 41011	1·97122 36586	·062
·939	1·54643 31223	·12067 35595$_{5}$	7·78132 34603	·50549 59862	·12851 28430	1·97825 50747	·061
·940	1·55477 35946	·11912 29652	7·89100 57227	·50368 12163	·12672 65587	1·98538 27531	·060
·941	1·56322 36470	·11756 39758	8·00415 25763	·50185 44123	·12493 51497	1·99260 97598	·059
·942	1·57178 68165^{+}	·11599 64802	8··) 60702	·50001 51720$_{5}$	·12313 85140	1·99993 93136	·058
·943	1·58046 68184	·11442 03633	8·2, ,3 99938	·49816 30749	·12133 65464	2·00737 47942	·057
·944	1·58926 75570$_{5}$	·11283 55063	8·36616 08906	·49629 76799$_{5}$	·11952 91380	2·01491 97556	·056
·945	1·59819 31399	·11124 17865$_{5}$	8·49500 91988	·49441 85248	·11771 61762	2·02257 79372	·055
·946	1·60724 78919	·10963 90770	8·62831 05085^{+}	·49252 51242	·11589 75444	2·03035 32773	·054
·947	1·61643 63711	·10802 72462	8·76630 69580	·49061 69681	·11407 31216	2·03824 99283	·053
·948	1·62576 33863	·10640 61581	8·90925 87998	·48869 35207	·11224 27828	2·04627 22703	·052
·949	1·63523 40154	·10477 56715$_{5}$	9·05744 61226	·48675 42173	·11040 63978	2·05442 49323	·051
·950	1·64485 36270	·10313 56404	9·21117 08093	·48479 84636$_{5}$	·10856 38320	2·06271 28074	·050
·951	1·65462 79023	·10148 59128	9·37075 87012	·48282 56323	·10671 49451	2·07114 10766	·049
·952	1·66456 28611	·09982 63310	9·53656 20484	·48083 50613	·10485 95914	2·07971 52299	·048
·953	1·67466 48890	·09815 67313	9·70896 22581	·47882 60505^{+}	·10299 76195	2·08844 10924	·047
·954	1·68494 07677	·09647 69433	9·88837 29856	·47679 78588$_{5}$	·10112 88714	2·09732 48546$_{5}$	·046
·955	1·69539 77100	·09478 67895^{-}	10·07524 36616	·47474 97013	·09925 31827	2·10637 30998	·045
·956	1·70604 33967	·09308 60850^{-}	10·27006 34581	·47268 07449	·09737 03818	2·11559 28409	·044
·957	1·71688 60181$_{5}$	·09137 46371	10·47336 58131	·47059 01044$_{5}$	·09548 02895^{+}	2·12499 15596	·043
·958	1·72793 43222	·08965 22444	10·68573 35964	·46847 68383$_{5}$	·09358 27186	2·13457 72476	·042
·959	1·73919 76650$_{5}$	·08791 86967	10·90780 50013	·46633 99426	·09167 74731	2·14435 84571	·041
·960	1·75068 60710	·08617 37741	11·14028 03288	·46417 83470	·08976 43480	2·15434 43514$_{5}$	·040
·961	1·76241 02977	·08441 72460	11·38392 97729	·46199 09065^{-}	·08784 31280	2·16454 47690	·039
·962	1·77438 19102$_{5}$	·08264 88710	11·63960 24344	·45977 63955^{+}	·08591 35873	2·17497 02893	·038
·963	1·78661 33654	·08086 83956	11·90823 67514	·45753 34993	·08397 54886	2·18563 23123$_{5}$	·037
·964	1·79911 81067	·07907 55532	12·19087 26685$_{5}$	·45526 08050$_{5}$	·08202 85821	2·19654 31438	·036
·965	1·81191 06729	·07727 00634	12·48866 58227	·45295 67915	·08007 26046	2·20771 60974	·035
·966	1·82500 68211	·07545 16306$_{5}$	12·80290 42138	·45061 98170$_{5}$	·07810 72781	2·21916 56074	·034
·967	1·83842 36691	·07361 99429	13·13502 78467	·44824 81064$_{5}$	·07613 23091	2·23090 73605^{-}	·033
·968	1·85217 98586$_{5}$	·07177 46702	13·48665 20044	·44583 97357	·07414 73866	2·24295 84444	·032
·969	1·86629 57434	·06991 54633	13·85959 49232	·44339 26136	·07215 21809	2·25533 75253	·031
·970	1·88079 36081	·06804 19514	14·25591 09446	·44090 44622	·07014 63417	2·26806 50475^{-}	·030
·971	1·89569 79240	·06615 37406	14·67793 03948	·43837 27924	·06812 94960	2·28116 34692	·029
·972	1·91103 56476$_{5}$	·06425 04111	15·12830 78674	·43579 48768	·06610 12460	2·29465 75399	·028
·973	1·92683 65733	·06233 15149	15·61008 10466	·43316 77166	·06406 11664	2·30857 46275^{-}	·027
·974	1·94313 37511$_{5}$	·06039 65726	16·12674 29263	·43048 80042	·06200 88014	2·32294 51001	·026
·975	1·95996 39846	·05844 50698	16·68233 10053	·42775 20771	·05994 36613	2·33780 27919	·025
·976	1·97736 84283	·05647 64533	17·28153 84696	·42495 58640	·05786 52185	2·35318 55534	·024
·977	1·99539 33102	·05449 01262	17·92985 38697	·42209 48199	·05577 29030	2·36913 59215^{-}	·023
·978	2·01409 08121	·05248 54425^{+}	18·63373 82856	·41916 38469	·05366 60967	2·38570 19334	·022
·979	2·03352 01492$_{5}$	·05046 17007	19·40085 22490	·41615 71984	·05154 41274	2·40293 81298	·021
·980	2·05374 89105^{-}	·04841 81359	20·24034 96517	·41306 83602	·04940 62611	2·42090 67947$_{5}$	·020

$\frac{1}{2}(1+\alpha_x)$	x	z	$\frac{\frac{1}{2}(1+\alpha_x)}{z}$	$\frac{\frac{1}{2}(1-\alpha_x)}{z}$	$\frac{z}{\frac{1}{2}(1+\alpha_x)}$	$\frac{z}{\frac{1}{2}(1-\alpha_x)}$	$\frac{1}{2}(1-\alpha_x)$
·980	2·05374 89105^{-}	·04841 81359	20·24034 96517	·41306 83602	·04940 62611	2·42090 67947^{5}	·020
·981	2·07485 47343	·04635 39107	21·16326 29328	·40988 99039	·04725 16929	2·43967 95103	·019
·982	2·09692 74292	·04426 81043	22·18301 45166	·40661 33007	·04507 95359	2·45933 91270	·018
·983	2·12007 16897	·04215 96988	23·31610 58315^{-}	·40322 86868	·04288 88086	2·47998 22848	·017
·984	2·14441 06210	·04002 75629	24·58306 04688	·39972 45605^{-}	·04067 84176	2·50172 26833	·016
·985	2·17009 03776^{5}	·03787 04310	26·00973 83286	·39608 73857	·03844 71380	2·52469 53982	·015
·986	2·19728 63766	·03568 68772	27·62920 37219^{5}	·39230 10670^{5}	·03619 35874	2·54906 26562	·014
·987	2·22621 17693	·03347 52823	29·48444 14167	·38834 62395	·03391 61928	2·57502 17157	·013
·988	2·25712 92445^{5}	·03123 37903	31·63240 80738	·38419 92877	·03161 31481	2·60281 58617	·012
·989	2·29036 78779	·02896 02511	34·15025 63952	·37983 09609	·02928 23570	2·63275 00991	·011
·990	2·32634 78740	·02665 21422	37·14523 17976	·37520 43616	·02692 13558	2·66521 42202	·010
·991	2·36561 81268	·02430 64606	40·77105 32107	·37027 19262	·02452 72055^{-}	2·70071 78486	·009
·992	2·40891 55459	·02191 95666	45·25636 91596	·36497 07190	·02209 63373	2·73994 58309	·008
·993	2·45726 33903	·01948 69510	50·95717 64741	·35921 47385^{-}	·01962 43212	2·78385 01399	·007
·994	2·51214 43279	·01700 28705^{+}	58·46071 69201	·35288 15911	·01710 55035^{+}	2·83381 17525^{-}	·006
·995	2·57582 93035^{+}	·01445 97430	68·81173 46310	·34578 76112	·01453 24051	2·89194 86054	·005
·996	2·65206 98079	·01184 70585^{+}	84·07150 14744	·33763 65521	·01189 46371	2·96176 46364^{5}	·004
·997	2·74778 13854	·00914 91911	108·97138 1780	·32789 78389	·00917 67213	3·04973 03779	·003
·998	2·87816 17391	·00634 01932	157·40845 1465+	·31544 77985^{+}	·00635 28990	3·17009 66203	·002
·999	3·09023 23062	·00336 70901	296·69535 9238	·29699 23516	·00337 04605^{+}	3·36709 00771	·001

Note. We believe that x and z may be taken as correct to the figures tabled. They were worked of course to more figures than are shown. The possibility of error in the ratio $\frac{1}{2}(1+\alpha_x)/z$ is greater, and may amount to five units in the tenth decimal. It seemed better to leave the last two figures standing with this warning rather than destroy the symmetry of the table by cutting them out. We feel compelled however to show only twelve figures in the last three entries of this ratio.

A more extended system of symbols than heretofore has been adopted in this table to indicate the nature of the last figure. 5^{+} and 5^{-} signify as usual that the real number exceeds 5 and falls short of 5. The symbol 5^{e} denotes that the number is exactly 5 *to the extent of the calculations*, i.e. ·63719,16745^{e} denotes that x for $\frac{1}{2}(1+\alpha_x) = ·738$ was found to be ·63719,16745,00. It does not necessarily indicate that the value terminated at the tenth or twelfth decimal. Another innovation has been made. Consider ·60075,97742^{5}; the usual interpretation of this would be that the number as actually worked was terminated by 5, 50 or 500 as the case might be, and the computer was unable to settle whether to enter it as ·60075,97742 or ·60075,97743. In the present table there may be doubt as to the correctness of the twelfth figure and the affixed 5 has been used when the final figures are 48, 49, 50, 51 or 52. Thus ·60075,97742,48 or ·60075,97742,51 would not be printed as usual ·60075,97742 and ·60075,97743, but as ·60075,97742^{5}, precisely as ·60075,97742,50 is written ·60075,97742^{5}. This seems safer when we cannot be sure of one or two units in the twelfth decimal place, and is more accurate when the 5 is actually put on the machine in computing.

We have to thank most heartily Dr W. F. Sheppard for the original loan to the Laboratory of his twelve figure tables of $\frac{1}{2}(1+\alpha_x)$ to argument x, and more recently for extracts ($x = 2·1$ to $3·1$) from his sixteen figure table of $\log_e \frac{1}{2}(1-\alpha_x)$ to argument x by intervals of ·1. We have also to thank Mr Frank Robbins for determining a large number of the values of x.

THE DURATION OF PLAY.

By E. C. FIELLER, B.A.

§ 1. The Problem and its Equations.

Two persons, A and B, play at a game in which their chances of winning are respectively p and q, where

$$p+q=1.$$

A starts playing with a counters, B with b counters, and after each game the loser gives the winner one counter. The set finishes when one of the players loses his last counter.

These are the conditions of the problem that we propose to discuss.

Let $p_{m,n}$ be the chance that after n games B will hold m counters.

If $1<m<a+b-1$, B must either hold $(m+1)$ counters after the $(n-1)$st game, and lose the nth, the probability of which is $p\times p_{m+1,n-1}$, or else hold $(m-1)$ counters after the $(n-1)$st game, and win the nth, the probability of which is $q\times p_{m-1,n-1}$.

If $m=0$ or 1, B must hold $(m+1)$ counters after the $(n-1)$st game, and lose the nth; if $m=(a+b-1)$ or $(a+b)$, he must hold $(m-1)$ counters after the $(n-1)$st game, and win the nth.

Thus

$$(1)\qquad p_{m,n}=\begin{cases} p\cdot p_{m+1,n-1}+q\cdot p_{m-1,n-1}, & (1<m<a+b-1)\\ p\cdot p_{m+1,n-1}, & m=0,\ 1\\ q\cdot p_{m-1,n-1}. & m=a+b-1,\ a+b \end{cases}$$

Since B starts playing with b counters,

$$(1{\cdot}1)\qquad p_{b,0}=1.$$

In n games B can win or lose only n, or $(n-2)$, or $(n-4)$, ... counters, and the number of counters he holds must always lie between 0 and $(a+b)$. Thus

$$(1{\cdot}2)\qquad p_{b+n-2i+1,n}=0,\quad (i=1,\ 2,\ 3,\ \dots\ n)$$

and

$$(1{\cdot}3)\qquad p_{m,n}=0$$

unless $\qquad \mathrm{Max}\,((b-n),\ 0)\leqslant m\leqslant \mathrm{Min}\,((b+n),\ (a+b)).$

(1), (1·1), (1·2), and (1·3) are the equations of the problem. Instead of solving them analytically, we may regard them as defining a quantity $p_{m,n}$ whose values we

can set out in a diagram somewhat similar to Pascal's Arithmetic Triangle, which gives the values of the quantity ${}_nC_r$ defined by the equations

$${}_nC_r = {}_{n-1}C_r + {}_{n-1}C_{r-1};\ {}_0C_0 = 1;\ {}_nC_r = 0 \text{ unless } 0 \leqslant r \leqslant n.$$

We take a rectangular array of $(a+b+1)$ rows of points, and starting from the bottom, number the rows 0, 1, 2, ... $(a+b)$, and starting from the left, number the columns 0, 1, 2, 3, ... Against the point (n, m) common to row m and column n we write the value of $p_{m,n}$. It is not necessary to take more than $(a+b+1)$ rows, by virtue of (1·3).

§ 2. The Solution when a is Infinite.

(*a*) *The Individual Probabilities.*

For simplicity's sake we consider first the case in which A's fortune is unlimited, and so, consequently, is the number of rows. By (1·1) and (1·3), the number at $(0, b)$ will be 1, and all other entries in column 0 will be zero. By (1·2) and (1·3), the entries in column n will be zero above the $(b+n)$th and below the $(b-n)$th rows, and also in the $(b+n-1)$st, $(b+n-3)$rd, ... $(b-n+1)$st rows. By (1), the entry at (n, m) is obtained, for $m > 1$, by adding p times the entry at $(n-1, m+1)$ to q times the entry at $(n-1, m-1)$, and for $m = 0, 1$, by taking p times the entry at $(n-1, m+1)$. Hence (Figure 1*) until n becomes greater than b, the non-vanishing terms in column n are, reading upwards, exactly the successive terms of the binomial expansion

$$(p+q)^n.$$

	•	•	•	•	..	..	..	•
$b+3$	•	•	•	q^3	..	..	..	${}_bC_5 p^{b-5} q^5$
$b+2$	•	•	q^2	0	..	..	..	0
$b+1$	•	q	0	$3pq^2$	..	..	..	${}_bC_4 p^{b-4} q^4$
b	1	0	$2pq$	0	..	..	..	0
$b-1$	•	p	0	$3p^2q$	..	..	..	${}_bC_3 p^{b-3} q^3$
$b-2$	•	•	p^2	0	..	..	..	0
$b-3$	•	•	•	p^3	..	..	..	${}_bC_2 p^{b-2} q^2$
	•	•	•	•	..	..	..	0
2	•	•	•	•	..	..	..	${}_bC_1 p^{b-1} q$
1	•	•	•	•	..	..	..	0
0	•	•	•	•	..	..	..	p^b
	0	1	2	3	..	..	..	b

Figure 1.

For $n > b$, these binomial terms become modified on account of the modification for $m = 0, 1$ in the rule for forming the entries in row m. The terms in the columns

* The figure is drawn for $b=7$; if b has not this value the last column will stand, but the upper portion of it will not fall on the horizontal rows marked by the left-hand scale, as more rows will occur between 2 and $b-3$.

immediately succeeding the bth are easily seen to be those shown in Figure 2, and generalising from these we obtain for the entries in column $(b+2i)$ the values shown in the first column of Figure 3; from this column the next three columns are formed in accordance with equation (1). Since the entries in the last two columns of Figure 3 differ from those in the first two only by having $(i+1)$ in place of i, the correctness of the assumed values follows by induction.

Thus for $(n+m-b)$ even,

$$p_{m,n} = ({}_nC_{\frac{n+m-b}{2}} - {}_nC_{\frac{n-m-b}{2}})\, p^{\frac{n-m+b}{2}} q^{\frac{n+m-b}{2}}. \qquad (n \geqslant m+b > b) \tag{2·1}$$

$$p_{m,n} = {}_nC_{\frac{n+m-b}{2}}\, p^{\frac{n-m+b}{2}} q^{\frac{n+m-b}{2}}. \qquad (n \leqslant b, \text{ or } n < m+b > b) \tag{2·2}$$

$$p_{0,n} = ({}_nC_{\frac{n-b}{2}} - 2\,{}_{n-1}C_{\frac{n-b-2}{2}})\, p^{\frac{n+b}{2}} q^{\frac{n-b}{2}}. \qquad (n > b) \tag{2·3}$$

Equations (2·1), (2·2), (2·3) give the solution of equations (1), (1·1), (1·2), (1·3).

In Figures 2 and 3 the remaining terms of the unmodified binomial are shown below row 0 in rows numbered $-1, -2, -3, \ldots$. It will be seen that, for $m > 0$, the binomial coefficient at (n, m) is modified by the subtraction of that at $(n, -m)$; this result we can obtain directly*.

In any particular sequence of n games, at the end of which B has m counters (we suppose $(n+m-b)$ even), B must lose $\frac{1}{2}(n-m+b)$ times and win $\frac{1}{2}(n+m-b)$ times, so that the probability that this particular sequence will occur is

$$p^{\frac{n-m+b}{2}} \times q^{\frac{n+m-b}{2}}.$$

$p_{m,n}$ is this probability multiplied by the number of permissible sequences, and we can find this number by means of an elegant geometrical representation, used by Borel† to determine $p_{0,n}$. Any sequence of n games can be represented by a zigzag path of n steps starting at $(0, b)$ and finishing in column n, and going from (α, β) to $(\alpha+1, \beta+1)$ if B wins the $(\alpha+1)$st game, and to $(\alpha+1, \beta-1)$ if he loses it. For the sequence to last effectively n games, the representative path must never cross or touch row 0, except possibly at $(n, 0)$. Without this restriction, there are ${}_nC_{\frac{n+m-b}{2}}$ zigzag paths joining $(0, b)$ and (n, m), for this is the number of ways in which we can assign the positions of the $\frac{1}{2}(n+m-b)$ upward steps. Now consider any path from $(0, b)$ to (n, m) that comes into contact with row 0. It may have several points $(\nu_j, 0)$ in common with row 0; let $\nu = \text{Min}(\nu_j)$. Then if we substitute for that portion of the path that lies between $(\nu, 0)$ and (n, m) its reflection in row 0, we get a path joining $(0, b)$ and $(n, -m)$, and conversely. Thus the paths joining $(0, b)$ and (n, m) and intersecting row 0 are in one-to-one correspondence with the paths joining $(0, b)$ and $(n, -m)$, so that the number of the

* We have indicated the inductive proof, because we believe that it was by this method—which Laplace characterises somewhat contemptuously as "en quelque sort mécanique"—that De Moivre arrived at his results (see § 4).

† *Principes et Formules Classiques du Calcul des Probabilités*, 1925, Ch. v.

former is equal to the number of the latter, or ${}_nC_{\frac{n-m-b}{2}}$. Hence the number of permissible paths is

$$ {}_nC_{\frac{n+m-b}{2}} - {}_nC_{\frac{n-m-b}{2}}, $$

leading to (2·1), while by (1),

$$ p_{0,\,b+2i} = p \cdot p_{1,\,b+2i-1} = ({}_{b+2i-1}C_i - {}_{b+2i-1}C_{i-1})\, p^{b+i} q^i, $$

$$ \text{(2·31)} \qquad \text{i.e. } p_{0,\,b+2i} = \frac{b\,(b+2i-1)\,!}{i\,!\,(b+i)\,!}\, p^{b+i} q^i; $$

(2·3) and (2·31) are easily seen to be in agreement.

7	•	${}_{b+1}C_4 p^{b-3} q^4$	•	${}_{b+3}C_5 p^{b-2} q^5$	•
6	${}_bC_3 p^{b-3} q^3$	•	${}_{b+2}C_4 p^{b-2} q^4$	•	${}_{b+4}C_5 p^{b-1} q^5$
5	•	${}_{b+1}C_3 p^{b-2} q^3$	•	${}_{b+3}C_4 p^{b-1} q^4$	•
4	${}_bC_2 p^{b-2} q^2$	•	${}_{b+2}C_3 p^{b-1} q^3$	•	$({}_{b+4}C_4 - 1)\, p^b q^4$
3	•	${}_{b+1}C_2 p^{b-1} q^2$	•	$({}_{b+3}C_3 - 1)\, p^b q^3$	•
2	${}_bC_1 p^{b-1} q$	•	$({}_{b+2}C_2 - 1)\, p^b q^2$	•	$({}_{b+4}C_3 - {}_{b+4}C_1)\, p^{b+1} q^3$
1	•	$({}_{b+1}C_1 - 1)\, p^b q$	•	$({}_{b+3}C_2 - {}_{b+3}C_1)\, p^{b+1} q^2$	•
0	p^b	•	$({}_{b+2}C_1 - 2)\, p^{b+1} q$	•	$({}_{b+4}C_2 - 2\,{}_{b+3}C_1)\, p^{b+2} q^2$
−1		p^{b+1}	•	${}_{b+3}C_1 p^{b+2} q$	•
−2			p^{b+2}	•	${}_{b+4}C_1 p^{b+3} q$
−3				p^{b+3}	•
−4					p^{b+4}

Figure 2.

(*b*) *B's Chance of Ruin.*

Let ${}_nP_b$ be the chance that B will lose his last counter on or before the nth game. The chance that he will lose it on the $(b+2i+1)$st game being zero, we have, by (2·31),

$$ \text{(3)} \quad {}_{b+2i+1}P_b = {}_{b+2i}P_b = p^b \left[1 + b \cdot pq + \frac{b\,(b+3)}{2}\, p^2 q^2 + \ldots + \frac{b\,(b+2i-1)\,!}{i\,!\,(b+i)\,!}\, p^i q^i \right]. $$

It is convenient to transform this result by noting that B, if he has not previously lost all his counters, must have, at the end of $(b+2i)$ games, either 0, 2, 4, ... or $(2b+2i)$ counters, and at the end of $(b+2i+1)$ games, either 1, 3, 5, ... or $(2b+2i+1)$ counters. Thus

$$ \text{(3·1)} \qquad {}_{b+2i}P_b + p_{2,\,b+2i} + p_{4,\,b+2i} + \ldots + p_{2b+2i,\,b+2i} = 1, $$

whence

$$ \text{(3·11)} \qquad {}_{b+2i}P_b + \sum_{r=1}^{b+i} {}_{b+2i}C_{i+r}\, p^{b+i-r} q^{i+r} - \sum_{r=1}^{i} {}_{b+2i}C_{i-r}\, p^{b+i-r} q^{i+r} = 1, $$

or

$$ \text{(3·12)} \quad {}_{b+2i}P_b = \text{1st } (i+1) \text{ terms of } (p+q)^{b+2i} + \text{1st } i \text{ terms of } \left(\frac{p}{q}\right)^b (q+p)^{b+2i}. $$

Similarly

(3·2) $${}_{b+2i+1}P_b + p_{1,b+2i+1} + p_{3,b+2i+1} + \ldots + p_{2b+2i+1,b+2i+1} = 1,$$

whence

(3·21) $${}_{b+2i+1}P_b + \sum_{r=1}^{b+i+1} {}_{b+2i+1}C_{i+r}\, p^{b+i-r+1}q^{i+r} - \sum_{r=1}^{i+1} {}_{b+2i+1}C_{i-r+1}\, p^{b+i-r+1}q^{i+r} = 1,$$

or

(3·22)

$${}_{b+2i+1}P_b = \text{1st } (i+1) \text{ terms of } (p+q)^{b+2i+1} + \text{1st } (i+1) \text{ terms of } \left(\frac{p}{q}\right)^b (q+p)^{b+2i+1}.$$

6	$p^{b+i-3}q^{i+3}({}_{b+2i}C_{i+3} - {}_{b+2i}C_{i-3})$ •	$p^{b+i-2}q^{i+4}({}_{b+2i+2}C_{i+4} - {}_{b+2i+2}C_{i-2})$ •		
5	•	$p^{b+i-2}q^{i+3}({}_{b+2i+1}C_{i+3} - {}_{b+2i+1}C_{i-2})$ •	$p^{b+i-1}q^{i+4}({}_{b+2i+3}C_{i+4} - {}_{b+2i+3}C_{i-1})$	
4	$p^{b+i-2}q^{i+2}({}_{b+2i}C_{i+2} - {}_{b+2i}C_{i-2})$ •	$p^{b+i-1}q^{i+3}({}_{b+2i+2}C_{i+3} - {}_{b+2i+2}C_{i-1})$ •		
3	•	$p^{b+i-1}q^{i+2}({}_{b+2i+1}C_{i+2} - {}_{b+2i+1}C_{i-1})$ •	$p^{b+i}q^{i+3}({}_{b+2i+3}C_{i+3} - {}_{b+2i+3}C_i)$	
2	$p^{b+i-1}q^{i+1}({}_{b+2i}C_{i+1} - {}_{b+2i}C_{i-1})$ •	$p^{b+i}q^{i+2}({}_{b+2i+2}C_{i+2} - {}_{b+2i+2}C_i)$ •		
1	•	$p^{b+i}q^{i+1}({}_{b+2i+1}C_{i+1} - {}_{b+2i+1}C_i)$ •	$p^{b+i+1}q^{i+2}({}_{b+2i+3}C_{i+2} - {}_{b+2i+3}C_{i+1})$	
0	$p^{b+i}q^i({}_{b+2i}C_i - 2\,{}_{b+2i-1}C_{i-1})$ •	$p^{b+i+1}q^{i+1}({}_{b+2i+2}C_{i+1} - 2\,{}_{b+2i+1}C_i)$ •		
−1	•	$p^{b+i+1}q^i\,{}_{b+2i+1}C_i$ •	$p^{b+i+2}q^{i+1}\,{}_{b+2i+3}C_{i+1}$	
−2	$p^{b+i+1}q^{i-1}\,{}_{b+2i}C_{i-1}$ •	$p^{b+i+2}q^i\,{}_{b+2i+2}C_i$ •		
−3	•	$p^{b+i+2}q^{i-1}\,{}_{b+2i+1}C_{i-1}$ •	$p^{b+i+3}q^i\,{}_{b+2i+3}C_i$	
−4	$p^{b+i+2}q^{i-2}\,{}_{b+2i}C_{i-2}$ •	$p^{b+i+3}q^{i-1}\,{}_{b+2i+2}C_{i-1}$ •		
−5	•	$p^{b+i+3}q^{i-2}\,{}_{b+2i+1}C_{i-2}$ •	$p^{b+i+4}q^{i-1}\,{}_{b+2i+3}C_{i-1}$	
−6	$p^{b+i+3}q^{i-3}\,{}_{b+2i}C_{i-3}$ •	$p^{b+i+4}q^{i-2}\,{}_{b+2i+2}C_{i-2}$ •		
	$(p+q)^{b+2i}$	$(p+q)^{b+2i+1}$	$(p+q)^{b+2i+2}$	$(p+q)^{b+2i+3}$

Figure 3.

Now*

(3·3) $$\text{1st } s \text{ terms of } \{x+(1-x)\}^n = I_x(n-s+1,\, s)$$
$$= \int_0^x x^{n-s}(1-x)^{s-1}\,dx \Big/ \int_0^1 x^{n-s}(1-x)^{s-1}\,dx,$$

so that (3·22) gives

$${}_{b+2i+1}P_b = I_p(b+i+1,\, i+1) + \left(\frac{p}{q}\right)^b I_q(b+i+1,\, i+1),$$

* This theorem is due to Professor Pearson (*Biometrika*, Vol. XVI. p. 202). It can be proved thus: If m be integral, and $x = py$,

$$\int_0^p x^l(1-x)^m\,dx = \int_0^1 p^{l+1}y^l\{(1-p)+p(1-y)\}^m\,dy$$
$$= \sum_{r=0}^{m} p^{l+m-r+1}(1-p)^r \frac{m!}{r!\,(m-r)!} \frac{\Gamma(l+1)\,\Gamma(m-r+1)}{\Gamma(l+m-r+2)}$$
$$= \frac{\Gamma(l+1)\,\Gamma(m+1)}{\Gamma(l+m+2)} \sum_{r=0}^{m} \frac{(l+m+1)(l+m)\ldots(l+m-r+2)}{r!}\, p^{l+m-r+1}(1-p)^r$$
$$= \int_0^1 x^l(1-x)^m\,dx \times \text{sum of 1st } (m+1) \text{ terms of } \{p+(1-p)\}^{l+m+1}.$$

whence, using the first of equations (3),

$$(4) \qquad {}_nP_b = I_p\left(\frac{n+b+1}{2}, \frac{n-b+1}{2}\right) + \left(\frac{p}{q}\right)^b I_q\left(\frac{n+b+1}{2}, \frac{n-b+1}{2}\right),$$

or

$$I_p\left(\frac{n+b+2}{2}, \frac{n-b+2}{2}\right) + \left(\frac{p}{q}\right)^b I_q\left(\frac{n+b+2}{2}, \frac{n-b+2}{2}\right),$$

according as $n-b$ is odd or even.

Equation (4) expresses ${}_nP_b$ in terms of proportional areas under a Pearson's Type I Curve. The mode and mean of this curve are at, respectively*,

$$x = \frac{n+b-1}{2(n-1)} \text{ and } x = \frac{n+b+1}{2(n+1)} \text{ if } (n-b) \text{ be odd,}$$

$$x = \frac{n+b}{2n} \text{ and } x = \frac{n+b+2}{2(n+2)} \text{ if } (n-b) \text{ be even.}$$

As n increases, the curve becomes more and more nearly symmetrical, and the area under it is concentrated more and more closely round the mode, which tends to the limiting position $x = \frac{1}{2}$.

Thus

$$(5) \quad \operatorname*{Lt}_{n\to\infty} I_x\left(\frac{n+b+1}{2}, \frac{n-b+1}{2}\right) = \operatorname*{Lt}_{n\to\infty} I_x\left(\frac{n+b+2}{2}, \frac{n-b+2}{2}\right) = 0, \tfrac{1}{2}, \text{ or } 1,$$

according as $x < \frac{1}{2}$, $x = \frac{1}{2}$, or $x > \frac{1}{2}$.

Let P_b be the probability that B will ultimately be ruined. Then by (4) and (5),

$$(6) \qquad P_b = \operatorname*{Lt}_{n\to\infty} {}_nP_b = \left(\frac{p}{q}\right)^b \text{ if } q > \tfrac{1}{2},$$

$$= 1 \quad \text{if } q \leqslant \tfrac{1}{2}.$$

Anybody, therefore, who plays continually at a game in which he has not a definite advantage is morally certain† to be ruined.

(*c*) *The Value of B's Expectation.*

Let B's expectation, when it is agreed to limit the set to n games, be ${}_nE_b$. Then

$$(7) \qquad {}_nE_b = \sum_{m=1}^{b+n} m p_{m,n}.$$

If B has m counters, his expectation on the next game is $m+q-p$, and accordingly

$$(7{\cdot}1) \qquad {}_{n+1}E_b = \sum_{m=1}^{b+n} (m+q-p)\, p_{m,n} = {}_nE_b + (q-p)(1 - {}_nP_b).$$

If $p = q = \frac{1}{2}$, ${}_nE_b$ is constant from game to game, and equal to b.

* See p. 1 of "The Numerical Evaluation of the Incomplete B-Function," by H. E. Soper (*Tracts for Computers*, No. VII).

† "En représentant, comme on le fait ordinairement, par l'unité la certitude absolue, celle par exemple qui résulte d'une démonstration rigoureuse, on pourra regarder comme une certitude morale toute fraction variable qui, sans devenir jamais égale à l'unité, peut en approcher d'assez près pour surpasser toute fraction déterminée." AMPÈRE.

Accordingly, if a man embarks on a career of inveterate gambling at a fair game, his expectation at the outset is equal to the amount that he has decided to risk, and at any subsequent moment, to the amount that he has still in hand. This result does not contradict the fact that he is morally certain to lose it. The chance that he will have anything left after n games becomes infinitely small as n increases, but the amount that he can expect to have, if he has anything, becomes correspondingly infinitely great. The objection to inveterate gambling lies in the practical consideration, that it is not worth while to face a very strong risk of being penniless, for the sake of a very slight chance of becoming a millionaire.

These remarks apply only to the case of $q=\frac{1}{2}$. For B's expectation on the $(b+2i)$th game, when p and q are not equal, we have

$$
\begin{aligned}
{}_{b+2i}E_b &= 2p^{b+i-1}q^{i+1}\left\{\sum_{r=1}^{b+i} r\,{}_{b+2i}C_{i+r}\left(\frac{q}{p}\right)^{r-1} - \sum_{r=1}^{i} r\,{}_{b+2i}C_{i-r}\left(\frac{q}{p}\right)^{r-1}\right\} \\
&= 2p^{b+i-1}q^{i+1}\frac{d}{dt}\left(\sum_{r=1}^{b+i}{}_{b+2i}C_{i+r}\,t^r - \sum_{r=1}^{i}{}_{b+2i}C_{i-r}\,t^r\right) \qquad \left(t=\frac{q}{p}\right) \\
&= 2p^{b+i-1}q^{i+1}\frac{d}{dt}\left\{\frac{1}{p^{b+i}q^i}\left(1-{}_{b+2i}P_b\right)\right\} \text{ by (3·11)} \\
&= \frac{2t^{i+1}}{(1+t)^{b+2i}\,B(b+i+1,\,i+1)}\frac{d}{dt}\left\{\frac{(1+t)^{b+2i}}{t^i}B(b+i+1,\,i+1)\right. \\
&\quad \left. - \frac{(1+t)^{b+2i}}{t^i}\int_0^{\frac{1}{1+t}} x^{b+i}(1-x)^i\,dx - \frac{(1+t)^{b+2i}}{t^{b+i}}\int_0^{\frac{t}{1+t}} x^{b+i}(1-x)^i\,dx\right\} \text{ by (4)} \\
&= 2\left\{(b+2i)\,q-i\right\}\left\{1-I_p(b+i+1,\,i+1)\right\} \\
&\quad - 2\left(\frac{p}{q}\right)^b\left\{(b+2i)\,q-(b+i)\right\}I_q(b+i+1,\,i+1),
\end{aligned}
$$

or, by (4),

$$(7\cdot2)\qquad {}_{b+2i}E_b = 2\left\{bq+i(q-p)\right\}\left\{1-{}_{b+2i}P_b\right\} + 2\left(\frac{p}{q}\right)^b b\,.\,I_q(b+i+1,\,i+1),$$

whence, by (7·1),

$$(7\cdot3)\qquad {}_{b+2i+1}E_b = 2\left\{bq+(i+\tfrac{1}{2})(q-p)\right\}\left\{1-{}_{b+2i}P_b\right\} + 2\left(\frac{p}{q}\right)^b b\,.\,I_q(b+i+1,\,i+1).$$

For $q>\frac{1}{2}$, $\quad \underset{i\to\infty}{\mathrm{Lt}}\ {}_{b+2i}P_b = \left(\frac{p}{q}\right)^b$,

and ${}_{b+2i}E_b$ and ${}_{b+2i+1}E_b$ tend to infinity with

$$i(q-p)\left(1-{}_{b+2i}P_b\right).$$

For $q<\frac{1}{2}$, $\quad \underset{i\to\infty}{\mathrm{Lt}}\left(1-{}_{b+2i}P_b\right) = \underset{i\to\infty}{\mathrm{Lt}}\ I_q(b+i+1,\,i+1) = 0,$

and accordingly

$$\underset{i\to\infty}{\mathrm{Lt}}\ {}_{b+2i}E_b = \underset{i\to\infty}{\mathrm{Lt}}\ {}_{b+2i+1}E_b = \underset{i\to\infty}{\mathrm{Lt}}\ i(q-p)\left(1-{}_{b+2i}P_b\right) = 0,$$

since $i(q-p)\left(1-{}_{b+2i}P_b\right)$ is negative for $q<\frac{1}{2}$, while the expectations are essentially positive.

Denoting by E_b B's expectation at the outset, when no limit is imposed on the number of games, we have, therefore,

(8) $\qquad E_b = 0,\ b,$ or ∞ according as $q<,\ =,$ or $>\frac{1}{2}$.

(*d*) *The Probable Length of the Set.*

If q be greater than p, there is a finite probability that B will never be ruined and the probable length of the set is infinite.

If $q \leqslant p$, the probable number of games is

$$\sum_{i=0}^{\infty} (b+2i)\, p_{0,\, b+2i} = bp^b \sum_{i=0}^{\infty} \frac{(b+2i)!}{i!\,(b+i)!}\, p^i q^i.$$

The ratio of the ith and $(i+1)$st terms of this series is

$$\frac{1}{pq}\,\frac{i(b+i)}{(b+2i-1)(b+2i)} = \frac{1}{4pq}\left\{1 + \frac{1}{2i} + O\left(\frac{1}{i^2}\right)\right\};$$

for $p=q$, $4pq=1$, and the series therefore diverges, so that the probable length of the set is again infinite.

For $q<p$, the series converges to the limit

$$\begin{aligned}\sum_{i=0}^{\infty} (b+2i)\, p_{0,\, b+2i} &= bP_b + 2pq\,\frac{d}{d\,(pq)}\,\frac{P_b}{p^b} \\ &= b + 2pq\,\frac{d\,(p)}{d\,(pq)}\cdot\frac{d}{dp}\cdot\frac{1}{p^b} \\ &= b\left[1 + \frac{2q}{p^b\,(p-q)}\right];\end{aligned}$$

this, when $q<p$, is the probable number of games that will be played before B is ruined.

§ 3. The Solution when a and b are Finite.

(*a*) *The Individual Probabilities.*

We proceed to the general case, in which A, as well as B, starts playing with a limited number of counters. Further modifications must now be made in our table of probabilities. In the case first considered, the set stopped if B lost b counters; now it stops also, if he wins a counters. The path representing any permissible sequence of games must now lie below row $(a+b)$, as well as above row 0. The binomial coefficients in our table, instead of being modified, as it were by reflection, only in row 0 after column b, will now be modified also, in the same way, in row $(a+b)$ after column a. Then, after column $(a+2b)$, (we suppose $b \leqslant a$), the modified coefficients will again be modified in row $(a+b)$, after column $(2a+b)$ and again after column $(2a+3b)$ in row 0, after column $(3a+2b)$ and again after column $(3a+4b)$ in row $(a+b)$, and so on.

Let V be the point $(0, b)$, and let $A_1, A_2, A_3, \ldots, B_1, B_2, B_3, \ldots$ be its two sets of reflections in row 0 and row $(a+b)$, so that $A_1, A_2, A_3, \ldots$ are the points $(0, 2a+b)$, $(0, -(2a+b))$, $(0, 4a+3b), \ldots$ and $B_1, B_2, B_3, \ldots$ the points $(0, -b)$, $(0, 2a+3b)$, $(0, -(2a+3b)), \ldots$.

We have seen that, for $n \leqslant b$, the entries in column b are the terms of the binomial expansion $(p+q)^n$. We call this array, starting from V, of the terms of the successive powers of $(p+q)$, the binomial table $\{(p+q)^n, V\}$. For rows above row 0, the modification of table $\{(p+q)^n, V\}$ in row 0 consists in the introduction

of the table $\left\{-\left(\frac{p}{q}\right)^{b}(q+p)^{n}, B_1\right\}$; accordingly, for rows between row 0 and row $(a+b)$, the modification of these two tables in row $(a+b)$ will consist in the introduction of tables $\left\{-\left(\frac{q}{p}\right)^{a}(p+q)^{n}, A_1\right\}$, $\left\{\left(\frac{q}{p}\right)^{a+b}(p+q)^{n}, B_2\right\}$, the modification of these, in row 0, in the introduction of tables

$$\left\{\left(\frac{p}{q}\right)^{a+b}(q+p)^{n}, A_2\right\}, \quad \left\{-\left(\frac{p}{q}\right)^{a+2b}(q+p)^{n}, B_3\right\},$$

and so on (see Figure 4).

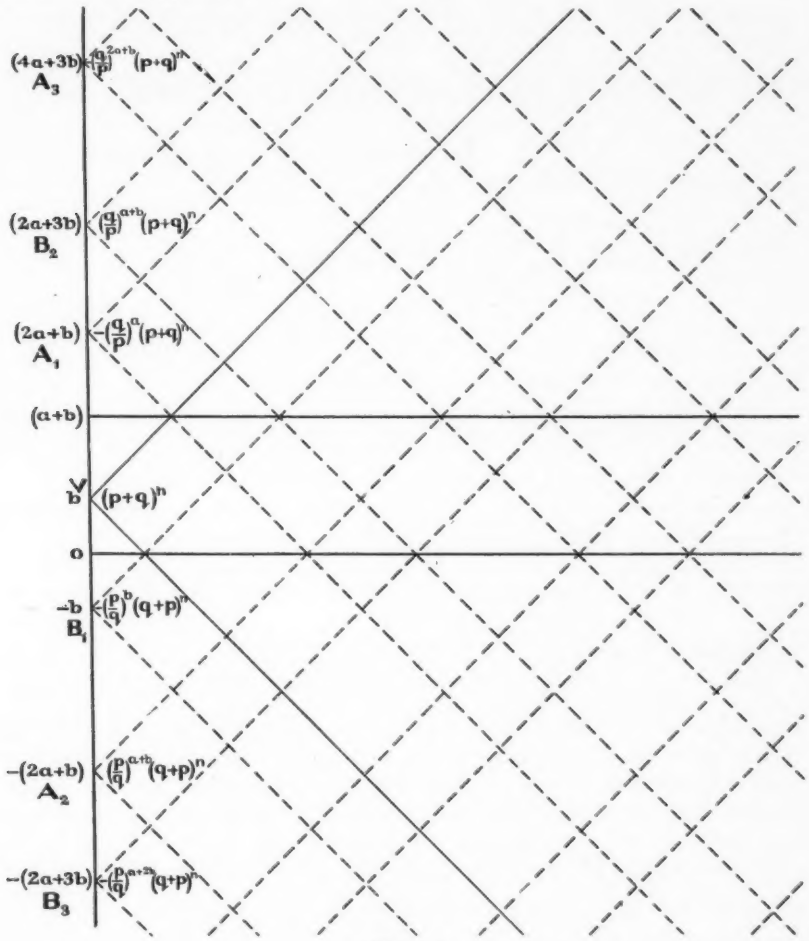


Figure 4.

For the entry at $(b+2i, 0)$ we have, associated with the tables whose vertices are at V and B_1, a term

$$\frac{b(b+2i-1)!}{i!\,(b+i)!}p^{b+i}q^i,$$

with the tables vertices A_1 and A_2 a term $\dfrac{-(b+2a)(b+2i-1)!}{(i-a)!\,(a+b+i)!}p^{b+i}q^i$,

" " " B_2 and B_3 " $\dfrac{+(3b+2a)(b+2i-1)!}{(i-a-b)!\,(a+2b+i)!}p^{b+i}q^i$,

and so on.

The probability that B will lose his last counter on the $(b+2i)$th game is therefore

$$(9{\cdot}1)\quad p_{0,\,b+2i}=(b+2i-1)!\left\{\sum_{r=0}^{r(a+b)\leqslant i}\frac{b+2r(a+b)}{(i-r(a+b))!\,(b+r(a+b)+i)!}-\sum_{r=1}^{r(a+b)\leqslant i+b}\frac{2r(a+b)-b}{(i-r(a+b)+b)!\,(r(a+b)+i)!}\right\}p^{b+i}q^i,$$

a result which we may write

$$(9{\cdot}2)\quad p_{0,\,n}=\left\{\sum_{r=0}^{r(a+b)\leqslant\frac{n-b}{2}}\frac{b+2r(a+b)}{n}\,{}_nC_{\frac{n-b-2r(a+b)}{2}}-\sum_{r=1}^{r(a+b)\leqslant\frac{n+b}{2}}\frac{a+(2r-1)(a+b)}{n}\,{}_nC_{\frac{n+b-2r(a+b)}{2}}\right\}p^{\frac{n+b}{2}}q^{\frac{n-b}{2}},$$

if $n-b$ be even and non-negative, and zero otherwise.

The probability that B will have m counters left after n games have been played is zero if $(n+m-b)$ be odd; if $(n+m-b)$ be even, it is, for $0<m<a+b$,

$$(9{\cdot}3)\quad p_{m,\,n}=\left[\sum_{r=0}^{2r(a+b)\leqslant n+m-b}{}_nC_{\frac{n+m-b-2r(a+b)}{2}}-\sum_{r=0}^{2r(a+b)\leqslant n-m-b}{}_nC_{\frac{n-m-b-2r(a+b)}{2}}-\sum_{r=1}^{2r(a+b)\leqslant n+m+b}{}_nC_{\frac{n+m+b-2r(a+b)}{2}}+\sum_{r=1}^{2r(a+b)\leqslant n-m+b}{}_nC_{\frac{n-m+b-2r(a+b)}{2}}\right]p^{\frac{n-m+b}{2}}q^{\frac{n+m-b}{2}}.$$

(*b*) *Ellis's Theorem.*

Since ${}_nC_{\frac{n+m-b}{2}}={}_nC_{\frac{n-m+b}{2}}$, the second member of (9·3) is unaltered by interchanging p with q, b with m, and a with $(a+b-m)$; thus in the course of the set, during which the sum of A's and B's possessions must remain constant, the probability, when B holds b counters, that after n more games he will hold m counters, is the same as the probability, when A holds m counters, that after n more games he will hold b counters. This symmetry is readily explained by the geometrical representation; for if we take any path permissible in one case and reflect it first in the line $x=\dfrac{n}{2}$ and then in the line $y=\dfrac{a+b}{2}$, we arrive at a path permissible in the other case.

By (2·1), a like symmetry exists when a is infinite—a player B', whose chance of winning is equal to B's chance of losing, stands the same chance of having b counters after n games if he starts with m, as B does, of having m counters after n games if he starts with b. It is clear that if we use column $(n-r)$ in the table of B's chances as column r in a table of B''s chances, any path joining $(0, b)$ and (n, m) represents corresponding sequences of games, and its chance of being described is the same, whether we are considering B's or B''s possessions.

(c) *Other Expressions for the Probabilities.*

We return to equation (9·3); it gives, for $(n+m-b)$ even, and $0 < m < a+b$:

$$p_{m,n} = \left[\sum_{r=0}^{2r(a+b)\leqq n+m-b} {}_nC_{\frac{n+m-b-2r(a+b)}{2}} + \sum_{r=1}^{2r(a+b)\leqq n-m+b} {}_nC_{\frac{n+m-b+2r(a+b)}{2}} - \sum_{r=0}^{2r(a+b)\leqq n-m-b} {}_nC_{\frac{n-m-b-2r(a+b)}{2}} - \sum_{r=1}^{2r(a+b)\leqq n+m+b} {}_nC_{\frac{n-m-b+2r(a+b)}{2}} \right] p^{\frac{n-m+b}{2}} q^{\frac{n+m-b}{2}}$$

or

$$(9\cdot4) \qquad p_{m,n} = \left[\Sigma\, {}_nC_{\frac{n+m-b\pm 2r(a+b)}{2}} - \Sigma\, {}_nC_{\frac{n-m-b\pm 2r(a+b)}{2}} \right] p^{\frac{n-m+b}{2}} q^{\frac{n+m-b}{2}};$$

hence

$$(9\cdot5) \qquad p_{0,n} = p \cdot p_{1,n-1} = \left[\Sigma\, {}_{n-1}C_{\frac{n-b\pm 2r(a+b)}{2}} - \Sigma\, {}_{n-1}C_{\frac{n-2-b\pm 2r(a+b)}{2}} \right] p^{\frac{n+b}{2}} q^{\frac{n-b}{2}},$$

$$(9\cdot6) \qquad p_{a+b,n} = q \cdot p_{a+b-1,n-1} = \left[\Sigma\, {}_{n-1}C_{\frac{n-a\pm 2r(a+b)}{2}} - \Sigma\, {}_{n-1}C_{\frac{n-2-a\pm 2r(a+b)}{2}} \right] p^{\frac{n-a}{2}} q^{\frac{n+a}{2}}.$$

In (9·4), (9·5), and (9·6) the summations extend over all possible integral non-negative values of r.

Now the term independent of x in the expansion of $x^{-\frac{n+m-b}{2}}(1+x)^n$ is ${}_nC_{\frac{n+m-b}{2}}$ or zero, according as $(n+m-b)$ is even or odd. Accordingly, if we give x the $(a+b)$ values

$$\cos\frac{2r\pi}{a+b} + \sqrt{-1}\sin\frac{2r\pi}{a+b} \qquad (r = 0, 1, 2 \ldots (a+b-1)),$$

which are the $(a+b)$ several $(a+b)$th roots of unity, and add the corresponding values of $x^{-\frac{n+m-b}{2}}(1+x)^n$, we get $(a+b) \times \Sigma\, {}_nC_{\frac{n+m-b\pm 2r(a+b)}{2}}$ or zero, according as $(n+m-b)$ is even or odd. The sum is

$$\sum_{r=0}^{a+b-1} \left(\cos\frac{2r\pi}{a+b} + \sqrt{-1}\sin\frac{2r\pi}{a+b}\right)^{-\frac{n+m-b}{2}} \left(1 + \cos\frac{2r\pi}{a+b} + \sqrt{-1}\sin\frac{2r\pi}{a+b}\right)^n$$

$$= \sum_{r=0}^{a+b-1} \left(\cos\frac{b-m}{a+b}r\pi + \sqrt{-1}\sin\frac{b-m}{a+b}r\pi\right)\left(2\cos\frac{r\pi}{a+b}\right)^n.$$

The imaginary part must vanish, and

$$\frac{1}{a+b}\sum_{r=0}^{a+b-1}\cos\frac{b-m}{a+b}r\pi\left(2\cos\frac{r\pi}{a+b}\right)^n=\Sigma\,{}_nC_{\frac{n+m-b\pm 2r(a+b)}{2}}\quad\text{or}\quad 0;$$

similarly

$$\frac{1}{a+b}\sum_{r=0}^{a+b-1}\cos\frac{b+m}{a+b}r\pi\left(2\cos\frac{r\pi}{a+b}\right)^n=\Sigma\,{}_nC_{\frac{n-m-b\pm 2r(a+b)}{2}}\quad\text{or}\quad 0,$$

according as $(n+m-b)$ is even or odd. In the latter case $p_{m,n}$ vanishes, and the last two equations, with (9·4), (9·5), and (9·6), accordingly give

$$(9\cdot7)\qquad p_{m,n}=\frac{2^{n+1}p^{\frac{n-m+b}{2}}q^{\frac{n+m-b}{2}}}{a+b}\sum_{r=0}^{a+b-1}\sin\frac{br\pi}{a+b}\sin\frac{mr\pi}{a+b}\left(\cos\frac{r\pi}{a+b}\right)^n,$$

$$(9\cdot8)\qquad p_{0,n}=\frac{2^n p^{\frac{n+b}{2}}q^{\frac{n-b}{2}}}{a+b}\sum_{r=0}^{a+b-1}\sin\frac{br\pi}{a+b}\sin\frac{r\pi}{a+b}\left(\cos\frac{r\pi}{a+b}\right)^{n-1},$$

and

$$(9\cdot9)\qquad p_{a+b,n}=\frac{2^n p^{\frac{n-a}{2}}q^{\frac{n+a}{2}}}{a+b}\sum_{r=0}^{a+b-1}\sin\frac{br\pi}{a+b}\sin\frac{a+b-1}{a+b}r\pi\left(\cos\frac{r\pi}{a+b}\right)^{n-1}$$

(9·7) holding for all values of m in $0<m<a+b$.

In (9·7), (9·8), and (9·9), the sums may be taken from 1 to $(a+b-1)$, since the terms corresponding to $r=0$ vanish.

If we give to a any value larger that n, then equations (9·4) and (9·5) reduce to (2·1) and (2·3) respectively; (9·4) and (9·5) thus contain the solution to the problem, both when a is finite and when a is infinite. The equivalent equations, (9·7) and (9·8), accordingly give, in an infinite number of forms, the solution to the case dealt with in § 2.

(*d*) *B's chance of Ruin.*

As before, let ${}_nP_b$ be B's chance of losing his last counter on or before the nth game. Then

$$(10)\qquad {}_{b+2i+1}P_b={}_{b+2i}P_b=\sum_{j=0}^{i}p_{0,b+2j},$$

where $p_{0,p+2j}$ is given by (9·1).

By (3) and (3·22) we have

$$\sum_{j=0}^{i}\frac{b(b+2j-1)!}{(b+j)!\,j!}p^{b+j}q^j=\text{1st }(i+1)\text{ terms of }(p+q)^{b+2i+1}$$
$$+\text{1st }(i+1)\text{ terms of }\left(\frac{p}{q}\right)^b(q+p)^{b+2i+1}.$$

Thus (10) gives

$$(10{\cdot}1)\quad {}_{b+2i}P_b = {}_{b+2i+1}P_b$$

$$= \text{1st } (i+1) \text{ terms of } (p+q)^{b+2i+1} + \text{1st } (i+1) \text{ terms of } \left(\frac{p}{q}\right)^b (q+p)^{b+2i+1}$$

$$- \text{1st } (i-a+1) \ldots \left(\frac{q}{p}\right)^a (p+q)^{b+2i+1} - \text{1st } (i-a+1) \ldots \left(\frac{p}{q}\right)^{a+b} (q+p)^{b+2i+1}$$

$$+ \text{1st } (i-a-b+1) \ldots \left(\frac{q}{p}\right)^{a+b} (p+q)^{b+2i+1}$$

$$+ \text{1st } (i-a-b+1) \ldots \left(\frac{p}{q}\right)^{a+2b} (q+p)^{b+2i+1}$$

$$- \text{1st } (i-2a-b+1) \ldots \left(\frac{q}{p}\right)^{2a+b} (p+q)^{b+2i+1}$$

$$- \text{1st } (i-2a-b+1) \ldots \left(\frac{p}{q}\right)^{2a+2b} (q+p)^{b+2i+1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

or, by (3·3),

$$(10{\cdot}2)\quad {}_{b+2i}P_b = {}_{b+2i+1}P_b$$

$$= \sum_{r=0}^{r(a+b)\leqslant i} \left(\frac{q}{p}\right)^{r(a+b)} I_p(r(a+b)+b+i+1,\ i-r(a+b)+1)$$

$$- \left(\frac{q}{p}\right)^a \sum_{r=1}^{r(a+b)\leqslant i+b} \left(\frac{q}{p}\right)^{(r-1)(a+b)} I_p(r(a+b)+i+1,\ i+b-r(a+b)+1)$$

$$+ \left(\frac{p}{q}\right)^b \sum_{r=0}^{r(a+b)\leqslant i} \left(\frac{p}{q}\right)^{r(a+b)} I_q(r(a+b)+b+i+1,\ i-r(a+b)+1)$$

$$- \sum_{r=1}^{r(a+b)\leqslant i+b} \left(\frac{p}{q}\right)^{r(a+b)} I_q(r(a+b)+i+1,\ i+b-r(a+b)+1).$$

From (10·2) we can derive the well-known expressions for P_b, the chance that B will ultimately be ruined.

For if $q < \frac{1}{2}$,

$$\left(\frac{p}{q}\right)^b \sum_{r=0}^{r(a+b)\leqslant i} \left(\frac{p}{q}\right)^{r(a+b)} I_q(r(a+b)+b+i+1,\ i-r(a+b)+1)$$

$$= \int_0^q \sum_{r=0} \left(\frac{p}{q}\right)^{r(a+b)+b} \frac{(b+2i+1)(b+2i)!}{(r(a+b)+b+i)!\,(i-r(a+b))!} x^{r(a+b)+b+i}(1-x)^{i-r(a+b)}\,dx$$

$$< \int_0^q (b+2i+1)\left(\frac{q}{p}\right)^i \left(1-x+\frac{px}{q}\right)^{b+2i} dx = \frac{q}{p-q}\left(\frac{q}{p}\right)^i \left[\left(1+\frac{p-q}{q}\,x\right)^{b+2i+1}\right]_0^q$$

$$= \frac{q}{p-q}\left\{(2p)^{b+1}(4pq)^i - \left(\frac{q}{p}\right)^i\right\} \to 0 \text{ as } i \to \infty.$$

Similarly, the last sum in (10·2) tends to zero if $q < \frac{1}{2}$, and the first two sums tend to zero if $q > \frac{1}{2}$.

If $q > \frac{1}{2}$,

$$\operatorname*{Lt}_{i\to\infty}\left\{\left(\frac{p}{q}\right)^{r(a+b)} I_q(r(a+b)+b+i+1,\, i-r(a+b)+1)\right\} = \left(\frac{p}{q}\right)^{r(a+b)},$$

$$\left(\frac{p}{q}\right)^{r(a+b)} I_q(r(a+b)+b+i+1,\, i-r(a+b)+1) < \left(\frac{p}{q}\right)^{r(a+b)},$$

and $\sum_{r=1}^{\infty}\left(\frac{p}{q}\right)^{r(a+b)}$ converges to $\dfrac{1}{1-\left(\frac{p}{q}\right)^{a+b}}$.

Hence, by Tannery's Theorem *,

$$\sum_{r=0}^{r(a+b)\leqslant i}\left(\frac{p}{q}\right)^{r(a+b)} I_q(r(a+b)+b+i+1,\, i-r(a+b)+1) \to \frac{1}{1-\left(\frac{p}{q}\right)^{a+b}} \text{ as } i\to\infty.$$

Similarly, for $q > \frac{1}{2}$ the last sum in (10·2) tends to $\dfrac{\left(\frac{p}{q}\right)^{a+b}}{1-\left(\frac{p}{q}\right)^{a+b}}$, and for $q < \frac{1}{2}$ the first two sums tend to $\dfrac{1}{1-\left(\frac{q}{p}\right)^{a+b}}$ and $\dfrac{\left(\frac{q}{p}\right)^{a}}{1-\left(\frac{q}{p}\right)^{a+b}}$ respectively.

Thus for $q < \frac{1}{2}$,
$$P_b = \frac{1-\left(\frac{q}{p}\right)^{a}}{1-\left(\frac{q}{p}\right)^{a+b}},$$

and for $q > \frac{1}{2}$,
$$P_b = \frac{\left(\frac{p}{q}\right)^{b}-\left(\frac{p}{q}\right)^{a+b}}{1-\left(\frac{p}{q}\right)^{a+b}} = \frac{\left(\frac{q}{p}\right)^{a}-1}{\left(\frac{q}{p}\right)^{a+b}-1},$$

so that

(11·1)
$$P_b = \frac{1-\left(\frac{q}{p}\right)^{a}}{1-\left(\frac{q}{p}\right)^{a+b}} \qquad (p \neq q).$$

P_b must clearly be a decreasing function of $\frac{q}{p}$, so that its value for $p = q$ must lie between the limits to which P_b tends as $\frac{q}{p}$ tends to unity from above and from below. But these limits are both $\frac{a}{a+b}$. Accordingly

(11·2)
$$P_b = \frac{a}{a+b} \qquad (p = q).$$

The values given in (6) are the limits of those given in (11·1) and (11·2), when $a \to \infty$.

* See Bromwich, *Infinite Series*, p. 136.

§ 4. SOME APPROXIMATE RESULTS.

So far we have been concerned with exact results. B's chance of being ruined in a given number of games, when A's fortune is unlimited, is given very simply by equation (4), and until n and b become considerable its numerical value may be found from the Tables of the Incomplete Beta-Function shortly to be published. Outside the range of these tables, the integrals can be evaluated approximately, by Weddling, by Dr Müller's continued fraction, or by the methods dealt with by Dr Wishart*.

If $p=q=\frac{1}{2}$, B's chance of being ruined in n games is, if $n+b$ be even,

$$(12) \qquad {}_nP_b = 2I_{\frac{1}{2}}\left(\frac{n+b+2}{2}, \frac{n-b+2}{2}\right).$$

We proceed to examine some methods of approximating to this chance, when n is large compared with b.

Method A.

The mode and the mean of the Type I Curve

$$(12{\cdot}1) \qquad y = \frac{\Gamma(n+2)}{\Gamma\left(\frac{n+b+2}{2}\right)\Gamma\left(\frac{n-b+2}{2}\right)} x^{\frac{n+b}{2}}(1-x)^{\frac{n-b}{2}}$$

being at $\frac{1}{2}+\frac{b}{2n}$, $\frac{1}{2}+\frac{b}{2(n+2)}$ respectively, the median is at $\frac{1}{2}+\frac{b(3n+2)}{6n(n+2)}$ approximately; accordingly we write (12) in the form

$$\tfrac{1}{2}(1-{}_nP_b) = \frac{\Gamma(n+2)}{\Gamma\left(\frac{n+b+2}{2}\right)\Gamma\left(\frac{n-b+2}{2}\right)} \left\{\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{b}{2n}} x^{\frac{n+b}{2}}(1-x)^{\frac{n-b}{2}}\,dx \right.$$

$$\left. - \int_{\frac{1}{2}+\frac{b(3n+2)}{6n(n+2)}}^{\frac{1}{2}+\frac{b}{2n}} x^{\frac{n+b}{2}}(1-x)^{\frac{n-b}{2}}\,dx\right\}.$$

The second member of Dr Wishart's equation (27) (*Biometrika*, Vol. XIX. p. 29) gives an approximation to the modal integral† of a Type I Curve in a series of Incomplete Normal Moment Functions. Using his result we have

$$(13) \qquad \tfrac{1}{2}(1-{}_nP_b) = k_0\{M(u_1) - M(u_2)\},$$

where $$u_1 = \frac{b\sqrt{n}}{\sqrt{n^2-b^2}}, \quad u_2 = \frac{4}{3(n+2)}\frac{b\sqrt{n}}{\sqrt{n^2-b^2}},$$

$$k_0 = \left(1+\frac{1}{n}\right)\left\{1 - \frac{1}{12n}\left(\frac{3n^2+b^2}{n^2-b^2}\right) + \frac{1}{288n^2}\left(\frac{3n^2+b^2}{n^2-b^2}\right)^2\right\},$$

* "The Approximate Quadrature of Certain Skew Curves," *Biometrika*, Vol. XIX.

† Not, as Dr Wishart states, to the Incomplete Beta-Function.

and $$M(u) = m_0(u) + \frac{4b}{3\sqrt{n(n^2-b^2)}}\, m_3(u) - \frac{3}{4}\frac{n^2+3b^2}{n(n^2-b^2)}\, m_4(u)$$
$$+ \frac{32}{5}\frac{b(n^2+b^2)}{\{n(n^2-b^2)\}^{\frac{3}{2}}}\, m_5(u) + \frac{5}{6}\left\{\frac{4b^2}{n(n^2-b^2)} - \frac{3(n^4+10n^2b^2+5b^4)}{n^2(n^2-b^2)^2}\right\} m_6(u)$$
$$- 8\,\frac{b(n^2+3b^2)}{\{n(n^2-b^2)\}^{\frac{3}{2}}}\, m_7(u) + \frac{7}{32}\,\frac{15n^4+346n^2b^2+109b^4}{n^2(n^2-b^2)^2}\, m_8(u)$$
$$+ \frac{512}{27}\,\frac{b^3}{\{n(n^2-b^2)\}^{\frac{3}{2}}}\, m_9(u) - \frac{105}{4}\,\frac{b^2(n^2+3b^2)}{n^2(n^2-b^2)^2}\, m_{10}(u) + \frac{385}{18}\,\frac{b^4}{n^2(n^2-b^2)^2}\, m_{12}(u);$$

here $m_0(u) = \int_0^u \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\, dx$ and $m_3(u), m_4(u), \ldots$ are the 3rd, 4th, ... Incomplete Normal Moment Functions tabled in *Tables for Statisticians and Biometricians.*

This expansion is valid only when u_1 and u_2 are not much greater than 1, that is, roughly, when $n > b^2$. In this case, since the coefficients in $M(u)$ are given only as far as terms in $\frac{1}{n^2}$, we lose nothing in accuracy by taking

$$k_0 = 1 + \frac{3}{4}\frac{1}{n} - \left(\frac{7}{32} + \frac{b^2}{3n}\right)\frac{1}{n^2}, \tag{13·1}$$

and

$$M(u) = m_0(u) + \frac{4}{3}\frac{b}{\sqrt{n}}\frac{1}{n}\left(1 + \frac{1}{2}\frac{b^2}{n^2}\right) m_3(u) - \frac{3}{4}\frac{1}{n}\left(1 + \frac{4b^2}{n^2}\right) m_4(u)$$
$$+ \frac{32}{5}\frac{b}{\sqrt{n}}\frac{1}{n^2}\, m_5(u) + \frac{5}{6}\frac{1}{n^2}\left(\frac{4b^2}{n} - 3\right) m_6(u)$$
$$- 8\,\frac{b}{\sqrt{n}}\frac{1}{n^2}\, m_7(u) + \frac{105}{32}\frac{1}{n^2}\, m_8(u). \tag{13·2}$$

Method B.

If we replace the Type I Curve (12·1) by a normal curve of the same mean and standard deviation, $\frac{1}{2} + \frac{b}{2(n+2)}$ and $\frac{1}{2(n+2)}\sqrt{\frac{(n+2)^2-b^2}{n+3}}$ respectively, then instead of (12) we have the approximate result

$$\tfrac{1}{2}(1 - {}_nP_b) = \int_0^b \sqrt{\frac{n+3}{(n+2)^2-b^2}}\,\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\, dx. \tag{14}$$

Method C.

Bertrand* has pointed out that if we approximate to the factorials in (2·31) by means of Stirling's Theorem, and make $p = q = \frac{1}{2}$, then that equation becomes

$$p_{0,\,b+2i} = \frac{2b}{\sqrt{2\pi}\,(b+2i)^{\frac{3}{2}}}\, e^{-\frac{1}{2}\frac{b^2}{b+2i}}; \tag{15}$$

thus, approximately,

$$1 - {}_{b+2i}P_b = \sum_{i+1}^{\infty} p_{0,\,b+2i} = \frac{2b}{\sqrt{2\pi}}\int_{i+1}^{\infty} \frac{e^{-\frac{1}{2}\frac{b^2}{b+2i}}}{(b+2i)^{\frac{3}{2}}}\, di.$$

* *Calcul des Probabilités*, Ch. VI.

Writing $b+2i=n$, $nx^2=b^2$, we find

$$\frac{1}{2}(1-{}_nP_b)=\int_0^{\frac{b}{\sqrt{n+2}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\,dx. \tag{16}$$

Method D.

We have seen that (9·8) gives the value of $p_{0,n}$ when a is infinite, if in that equation we give a any value $>n$. Accordingly

$${}_nP_b=P_b-\sum_{n+1}^{\infty}p_{0,n}=P_b-\frac{(4pq)^{\frac{n+1}{2}}}{a+b}\left(\frac{p}{q}\right)^{\frac{b}{2}}\sum_{r=1}^{\frac{a+b}{2}-1}\frac{\sin\frac{r\pi}{a+b}\sin\frac{rb\pi}{a+b}\left(\cos\frac{r\pi}{a+b}\right)^n}{1-2\sqrt{pq}\cos\frac{r\pi}{a+b}}.$$

When a is large compared with b, we may replace the sum by an integral, and write

$${}_nP_b=P_b-\frac{(4pq)^{\frac{n+1}{2}}}{\pi}\left(\frac{p}{q}\right)^{\frac{b}{2}}\int_0^{\pi}\frac{\sin r\phi\sin rb\phi\cos^n\phi}{1-\sqrt{4pq}\cos\phi}\,d\phi.$$

Putting $\phi+\phi'=\pi$, we have, integrating separately over the ranges 0 to $\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ to π,

$$\int_{\frac{\pi}{2}}^{\pi}\frac{\sin\phi\sin b\phi\cos^n\phi}{1-\sqrt{4pq}\cos\phi}\,d\phi=\int_0^{\frac{\pi}{2}}\frac{\sin\phi'(-1)^{b-1}\sin b\phi'(-1)^n\cos^n\phi'}{1+\sqrt{4pq}\cos\phi'}\,d\phi'$$

$$=(-1)^{n+b-1}\int_0^{\frac{\pi}{2}}\frac{\sin\phi\sin b\phi\cos^n\phi}{1+\sqrt{4pq}\cos\phi}\,d\phi,$$

whence

$${}_nP_b=P_b-(4pq)^{\frac{n'+2}{2}}\left(\frac{p}{q}\right)^{\frac{b}{2}}\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\sin\phi\sin b\phi\cos^{n'+1}\phi}{1-4pq\cos^2\phi}\,d\phi, \tag{17}$$

where $n'=n-1$ or n, according as $n+b$ is odd or even.

For $p=q=\frac{1}{2}$, (17) becomes

$${}_nP_b=1-\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\cos^{n'+1}\phi}{\sin\phi}\sin b\phi\,d\phi. \tag{17·1}*$$

Let

$$\Phi=\int_0^{\frac{\pi}{2}}\frac{\cos^s\phi}{\sin\phi}\sin b\phi\,d\phi.$$

The integrand in Φ has its maximum value, b, at $\phi=0$, and if s is large decreases very rapidly in numerical value as ϕ increases. The value of the integral is therefore due almost entirely to the contribution of a small range of ϕ near $\phi=0$, and for this range we have, neglecting powers of ϕ above the fifth,

$$\begin{aligned}\log\frac{\cos^s\phi}{\sin\phi}&=s\log\left(1-\frac{\phi^2}{2}+\frac{\phi^4}{24}\right)-\log\phi-\log\left(1-\frac{\phi^2}{6}+\frac{\phi^4}{120}\right)\\&=-s\left(\frac{\phi^2}{2}-\frac{\phi^4}{24}\right)-s\frac{\phi^4}{8}+\left(\frac{\phi^2}{6}-\frac{\phi^4}{120}\right)+\frac{\phi^4}{72}-\log\phi\\&=-\tfrac{1}{2}\phi^2(s-\tfrac{1}{3})-\tfrac{1}{12}\phi^4(s-\tfrac{1}{15})-\log\phi\\&=-\tfrac{1}{2}\phi^2(s-\tfrac{1}{3})+\log\{1-\tfrac{1}{12}\phi^4(s-\tfrac{1}{15})\}-\log\phi,\end{aligned}$$

* The method by which we proceed to equation (19) is due to Laplace (*Théorie Analytique des Probabilités*).

so that, approximately,

$$\frac{\cos^s\phi}{\sin\phi} = \frac{e^{-\frac{1}{2}\phi^2(s-\frac{1}{3})}}{\phi}\{1-\tfrac{1}{12}\phi^4(s-\tfrac{1}{15})\}.$$

The second member of this equation decreases very rapidly as ϕ increases; accordingly, treating

$$\int_{\frac{\pi}{2}}^{\infty} \frac{e^{-\frac{1}{2}\phi^2(s-\frac{1}{3})}}{\phi}\{1-\tfrac{1}{12}(s-\tfrac{1}{15})\phi^4\}\sin b\phi\, d\phi$$

as negligible, we have, approximately,

$$\Phi = \int_0^{\infty} \frac{e^{-\frac{1}{2}\phi^2(s-\frac{1}{3})}}{\phi}\{1-\tfrac{1}{12}(s-\tfrac{1}{15})\phi^4\}\sin b\phi\, d\phi, \tag{18}$$

whence*

$$\frac{d\Phi}{db} = \int_0^{\infty} e^{-\frac{1}{2}\phi^2(s-\frac{1}{3})}\{1-\tfrac{1}{12}(s-\tfrac{1}{15})\phi^4\}\cos b\phi\, d\phi. \tag{18·1}$$

Now
$$\int_0^{\infty} e^{-\lambda^2\phi^2}\cos b\phi\, d\phi = \frac{\sqrt{\pi}}{2\lambda}e^{-\frac{b^2}{4\lambda^2}},$$

whence
$$\int^{\infty}\phi^4 e^{-\lambda^2\phi^2}\cos b\phi\, d\phi = \frac{\sqrt{\pi}}{2\lambda}\frac{d^4}{db^4}(e^{-\frac{b^2}{4\lambda^2}})$$
$$= \frac{3\sqrt{\pi}}{8\lambda^5}\left(1-\frac{b^2}{\lambda^2}+\frac{1}{12}\frac{b^4}{\lambda^4}\right)e^{-\frac{b^2}{4\lambda^2}}.$$

Thus we have from (18·1), writing $\frac{1}{2}(s-\frac{1}{3})=\lambda^2=\frac{1}{2}\sigma^2$,

$$\frac{d\Phi}{db} = \frac{\sqrt{\pi}}{2\lambda}\left\{1-\frac{1}{8\lambda^4}\left(\lambda^2+\frac{2}{15}\right)\left(1-\frac{b^2}{\lambda^2}+\frac{1}{12}\frac{b^4}{\lambda^4}\right)\right\}e^{-\frac{b^2}{4\lambda^2}}$$
$$= \frac{1}{\sigma}\sqrt{\frac{\pi}{2}}\left\{\left(1-\frac{1}{4\sigma^2}-\frac{1}{15\sigma^4}\right)+\frac{b^2}{\sigma^2}\left(\frac{1}{2\sigma^2}+\frac{2}{15\sigma^4}\right)-\frac{1}{6}\frac{b^4}{\sigma^4}\left(\frac{1}{2\sigma^2}+\frac{2}{15\sigma^4}\right)\right\}e^{-\frac{1}{2}\frac{b^2}{\sigma^2}}.$$

Now
$$\int_0^b \frac{b^2}{\sigma^2}e^{-\frac{1}{2}\frac{b^2}{\sigma^2}}db = -be^{-\frac{1}{2}\frac{b^2}{\sigma^2}}+\int_0^b e^{-\frac{1}{2}\frac{b^2}{\sigma^2}}db,$$
$$\int_0^b \frac{b^4}{\sigma^4}e^{-\frac{1}{2}\frac{b^2}{\sigma^2}}db = -\frac{b^3}{\sigma^2}e^{-\frac{1}{2}\frac{b^2}{\sigma^2}}-3be^{-\frac{1}{2}\frac{b^2}{\sigma^2}}+3\int_0^b e^{-\frac{1}{2}\frac{b^2}{\sigma^2}}db;$$

thus the equation

$$\Phi = \frac{1}{\sigma}\sqrt{\frac{\pi}{2}}\int_0^b\left\{\left(1-\frac{1}{4\sigma^2}-\frac{1}{15\sigma^4}\right)+\frac{b^2}{\sigma^2}\left(\frac{1}{2\sigma^2}+\frac{2}{15\sigma^4}\right)-\frac{1}{6}\frac{b^4}{\sigma^4}\left(\frac{1}{2\sigma^2}+\frac{2}{15\sigma^4}\right)\right\}e^{-\frac{1}{2}\frac{b^2}{\sigma^2}}db,$$

gives, after some reduction,

$$\Phi = \pi\int_0^b \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\frac{b^2}{\sigma^2}}db - \frac{\pi}{4}\frac{b}{\sigma^2}\left(1+\frac{4}{15\sigma^2}\right)\left(1-\frac{b^2}{3\sigma^2}\right)\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\frac{b^2}{\sigma^2}}. \tag{18·2}$$

* The differentiation under the integral sign is legitimate if the resulting integral is uniformly convergent; but this is so, for the integral obtained by omitting the trigonometrical factor in the integrand is absolutely convergent. A similar remark applies to the differentiations that follow.

In (18·2) take $s = n' + 1$, or $\sigma^2 = n' + \frac{2}{3}$; substitute in (17·1), and we have the approximate result

$$(19) \qquad \tfrac{1}{2}(1 - {}_nP_b) = \int_0^{\sqrt{n'+\frac{2}{3}}b} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx$$
$$- \frac{1}{4}\frac{b}{n'+\frac{2}{3}}\left(1 + \frac{4}{15}\frac{1}{n'+\frac{2}{3}}\right)\left(1 - \frac{1}{3}\frac{b^2}{n'+\frac{2}{3}}\right)\frac{1}{\sqrt{2\pi\,(n'+\frac{2}{3})}}\, e^{-\frac{1}{2}\frac{b^2}{n'+\frac{2}{3}}}.$$

(19) is a more accurate form of the equation given by Laplace (*loc. cit.* p. 259), which is, in our notation,

$$\tfrac{1}{2}(1 - {}_nP_b) = \int_0^{\sqrt{n+\frac{2}{3}}b} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx - \frac{1}{4}\frac{b}{n+\frac{2}{3}}\left(1 - \frac{1}{3}\frac{b^2}{n+\frac{2}{3}}\right)\frac{1}{\sqrt{2\pi\,(n+\frac{2}{3})}}\, e^{-\frac{1}{2}\frac{b^2}{n+\frac{2}{3}}}.$$

If we sum from b to ∞ the expression for $p_{0,n}$ given by (9·8), and then make a infinite, we get

$$P_b = \frac{(4pq)^{\frac{b}{2}}}{\pi}\left(\frac{p}{q}\right)^{\frac{b}{2}} \int_0^{\pi} \frac{\sin\phi \sin b\phi\,(\cos\phi)^{b-1}}{1 - \sqrt{4pq}\cos\phi}\,d\phi,$$

whence, as for equation (17),

$$(20) \qquad P_b = (2p)^b \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin\phi \sin b\phi\,(\cos\phi)^{b-1}}{1 - 4pq\cos^2\phi}\,d\phi.$$

If we substitute from (20) in (17), make $p = q$, and then approximate to the two integrals by means of (18·2), we arrive at

$$(21) \quad \tfrac{1}{2}\,{}_nP_b = \int_0^{\sqrt{b-\frac{4}{3}}b} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx - \int_0^{\sqrt{n'+\frac{2}{3}}b} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx$$
$$- \frac{1}{4}\frac{b}{b-\frac{4}{3}}\left(1 + \frac{4}{15}\frac{1}{b-\frac{4}{3}}\right)\left(1 - \frac{1}{3}\frac{b^2}{b-\frac{4}{3}}\right)\frac{1}{\sqrt{2\pi\,(b-\frac{4}{3})}}\, e^{-\frac{1}{2}\frac{b^2}{b-\frac{4}{3}}}$$
$$+ \frac{1}{4}\frac{b}{n'+\frac{2}{3}}\left(1 + \frac{1}{15}\frac{1}{n'+\frac{2}{3}}\right)\left(1 - \frac{1}{3}\frac{b^2}{n'+\frac{2}{3}}\right)\frac{1}{\sqrt{2\pi\,(n'+\frac{2}{3})}}\, e^{-\frac{1}{2}\frac{b^2}{n'+\frac{2}{3}}},$$

where, as in (19), $n' = n - 1$ or n, according as $n + b$ is odd or even.

Except for small values of b, (21) will not give results differing significantly from those provided by (19); for $b = 6$, $n = 78$, the true value of ${}_nP_b$ is ·499897; (19) gives ·499895, and (21), ·499710, so that (21) appears to be less accurate, as well as less simple, than (19).

Incidentally, by comparing (20) with (6), we have the analytical theorem

$$(22) \qquad \int_0^{\frac{\pi}{2}} \frac{\sin\phi \sin b\phi\,(\cos\phi)^{b-1}}{1 - \lambda\cos^2\phi}\,d\phi = (2\mu)^{-b}\frac{\pi}{2},$$

where $0 < \lambda \leqslant 1$, and μ is the greater root of

$$4\mu^2 - 4\mu + \lambda = 0.$$

We turn now to the converse problem: when B's fortune is known, A's unlimited, and $p=q$, what number of games must we assign to make B's chance of ruin assume any given value?

In general, of course, the question cannot, strictly speaking, be answered: all that we can hope to do is to find an integer n, even or odd with b, such that ${}_nP_b < P < {}_{n+2}P_b$, where P is the given value.

Approximately, n is given by the equation

$$I_{\frac{1}{2}}\left(\frac{n+b+2}{2}, \frac{n-b+2}{2}\right) = \tfrac{1}{2}P; \tag{23}$$

outside the range of the Incomplete Beta-Function Tables, approximations to n can be found by replacing (23) by equations (14) and (16), which can be solved by means of tables giving the abscissa of the normal curve in terms of its area*.

For $P=\frac{1}{2}$, the case discussed by Laplace (*loc. cit.* pp. 257—260), the problem is the same as that of finding a Type I Curve, with a given difference between its indices, and having one quartile at $x=\cdot5$. If, as in equation (14), we replace the Type I Curve by a normal curve having the same mean and standard deviation, we reach the approximate result

$$n+2 = \frac{b^2}{2t^2}\left(1+\sqrt{1+\frac{4t^2}{b^2}+\frac{4t^4}{b^2}}\right) \qquad (t=\cdot6744897502)$$

$$= 1\cdot09905467\, b^2\left(1+\sqrt{1+2\cdot64761429\, b^{-2}}\right),$$

or, slightly less exactly,

$$n = 2\cdot1981093\, b^2 - \cdot545064. \tag{24}$$

If we put ${}_nP_b = \frac{1}{2}$ in equation (13), take k_0 to be unity, neglect $M(u_2)$ because u_2 is small, and retain only the first term of $M(u_1)$, we get

$$\int_0^{\frac{b\sqrt{n}}{\sqrt{n^2-b^2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx = \tfrac{1}{4},$$

whence

$$n = \frac{b^2}{2t^2}\left(1+\sqrt{1+\frac{4t^4}{b^2}}\right),$$

or, approximately,

$$n = 2\cdot1981093\, b^2 + \cdot454936, \tag{25}$$

a result differing by 1 from that of (24).

If we put ${}_nP_b = \frac{1}{2}$ in (16), we have, at once,

$$n = 2\cdot1981093\, b^2 - 2. \tag{26}$$

Finally, we may apply the equation of Method D to the converse problem; there we take $n'=n$, because the value of n required is even or odd with b.

* *Tables for Statisticians and Biometricians*, Part I, Table III.

If, as a first approximation, we retain only the integral term in the second member of (19), we obtain, for the value of n which makes ${}_nP_b$ equal to a half,

$$\int_0^{\sqrt{n+\frac{2}{3}}b} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx = \tfrac{1}{4},$$

whence

$$n = 2{\cdot}1981093\,b^2 - \tfrac{2}{3} \text{ approximately,} \tag{27}$$

an equation not very different from (24), (25), and (26).

If we substitute this value of n in the remaining term in (19) we get

$$-\frac{1}{4}\frac{t^2}{b}\left(1+\frac{4}{15}\frac{t^2}{b^2}\right)\left(1-\frac{1}{3}t^2\right)\frac{t}{b}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2} \quad (t = {\cdot}6744897502)$$

$$= \frac{t^2-3}{12}\frac{t^3}{b^3}\left(1+\frac{4}{15}\frac{t^2}{b^2}\right) \times {\cdot}3177765727$$

$$= -{\cdot}0206807\,b^{-2} - {\cdot}0025089\,b^{-4}.$$

We have, therefore, for a second approximation to the value of n that makes ${}_nP_b$ equal to a half,

$$\int_0^{\sqrt{n+\frac{2}{3}}b} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx = {\cdot}25 + {\cdot}0206807\,b^{-2} + {\cdot}0025089\,b^{-4}. \tag{28}$$

We may solve this equation either by means of Table III in Part I of *Tables for Statisticians*, or by the following approximate process.

Let $$\alpha = 2\int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx,$$

so that $$\frac{dx}{d\alpha} = \frac{1}{2}\Big/\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\right), \qquad \frac{d^2x}{d\alpha^2} = \frac{1}{4}x\Big/\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\right)^2.$$

Take $\alpha = {\cdot}5$, so that* $$x = {\cdot}6744897502 = t,$$

and $$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} = {\cdot}3177765727 = \lambda \text{ say.}$$

Then for small ϵ, $$\alpha + \epsilon = 2\int_0^{t+\eta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx.$$

where $$t + \eta = t + \frac{1}{2\lambda}\epsilon + \frac{t}{8\lambda^2}\epsilon^2 \text{ approximately.}$$

Take $$\epsilon = \frac{3-t^2}{6}\frac{\lambda t^3}{b^2}\left(1+\frac{4}{15}\frac{t^2}{b^2}\right).$$

Then as far as terms in b^{-4},

$$t + \eta = t\left\{1 + \frac{3-t^2}{12}\frac{t^2}{b^2}\left(1+\frac{4}{15}\frac{t^2}{b^2}\right) + \frac{(3-t^2)^2}{288}\frac{t^6}{b^4}\right\},$$

* These values are taken from Kondo and Elderton's Tables of the Normal Curve.

so that (21) gives

$$n+\frac{2}{3}=\frac{b^2}{t^2}\left\{1+\frac{3-t^2}{12}\frac{t^2}{b^2}\left(1+\frac{4}{15}\frac{t^2}{b^2}\right)+\frac{(3-t^2)^2}{288}\frac{t^6}{b^4}\right\}^{-2}$$

$$=\frac{b^2}{t^2}-\frac{3-t^2}{6}-\frac{3-t^2}{6}\frac{t^2}{b^2}\left(\frac{4}{15}+\frac{3-t^2}{24}t^2-\frac{3-t^2}{8}\right)$$

$$=\frac{b^2}{t^2}-\frac{3-t^2}{6}-\frac{3-t^2}{6}\left\{\frac{4}{15}-\frac{(3-t^2)^2}{24}\right\}\frac{t^2}{b^2},$$

whence

(28·1) $$n=2{\cdot}1981093\,b^2-1{\cdot}090844+{\cdot}0006219\,b^{-2}.$$

In the table of numerical illustrations, the second and third columns show, for different values of b, the values of n provided by equations (24) and (28) respectively; the approximate constancy of the difference between these values indicates that for our purpose (23·1) is just as good as (28). To estimate the degree of accuracy to which (28·1) gives the value of n satisfying

(29) $$\int_0^{\frac{b^2}{\sqrt{n+\frac{2}{3}}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx-{\cdot}25$$

$$-\frac{1}{4}\frac{b}{n+\frac{2}{3}}\left(1+\frac{4}{15}\frac{1}{n+\frac{2}{3}}\right)\left(1-\frac{1}{3}\frac{b^2}{n+\frac{2}{3}}\right)\frac{1}{\sqrt{2\pi\,(n+\frac{2}{3})}}e^{-\frac{1}{2}\frac{b^2}{n+\frac{2}{3}}}=0$$

we write $\dfrac{b}{\sqrt{n+\frac{2}{3}}}=t+\eta$, where η will be small.

By Taylor's Theorem,

$$\int_0^{t+\eta}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx-{\cdot}25=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}\left(\eta-\frac{t}{2}\eta^2+\frac{t^2-1}{6}\eta^3+\frac{3t-t^3}{24}\eta^4+\ldots\right),$$

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(t+\eta)^2}=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}\left(1-t\eta+\frac{t^2-1}{2}\eta^2+\frac{3t-t^3}{6}\eta^3+\ldots\right),$$

and

$$\frac{1}{4}\frac{(t+\eta)^3}{b^2}\left\{1+\frac{4}{15}\frac{(t+\eta)^2}{b^2}\right\}\left\{1-\frac{1}{3}(t+\eta)^2\right\}=\frac{1}{4}\frac{t^3}{b^2}\left(\frac{3-t^2}{3}+\frac{4}{15}\frac{3-t^2}{3}\frac{t^2}{b^2}\right)$$

$$+\frac{1}{4}\frac{t^2}{b^2}\left(\frac{9-5t^2}{3}+\frac{4}{15}\frac{15-7t^2}{3}\frac{t^2}{b^2}\right)\eta+\frac{1}{4}\frac{t}{b^2}\left(\frac{9-10t^2}{3}+4\frac{10-7t^2}{15}\frac{t^2}{b^2}\right)\eta^2$$

$$+\ldots\ldots;$$

as far as terms in η^2, therefore,

$$\frac{1}{4}\frac{t^3}{b^2}\frac{3-t^2}{3}\left(1+\frac{4}{15}\frac{t^2}{b^2}\right)-\eta\left\{1-\frac{1}{4}\frac{t^2}{b^2}\left(\frac{9-8t^2+t^4}{3}+\frac{4}{15}\frac{15-10t^2+t^4}{3}\frac{t^2}{b^2}\right)\right\}$$

$$+\eta^2\frac{t}{2}\left\{1+\frac{1}{4b^2}\left(\frac{18-41t^2+14t^4-t^6}{3}+\frac{4}{15}\frac{60-75t^2+18t^4-t^6}{3}\frac{t^2}{b^2}\right)\right\}=0.$$

Neglecting terms in η^2 and $\dfrac{1}{b^4}$, we find from this equation

$$n=2{\cdot}1981093\,b^2-1{\cdot}090844;$$

solving the quadratic, we get, as far as terms in b^{-4},

$$\eta = \cdot 0650794 b^{-2} + \cdot 2439733 b^{-4},$$

whence

$$n = 2 \cdot 1981093 b^2 - 1 \cdot 090844 - 1 \cdot 528788 b^{-2}. \tag{30}$$

The terms that we have neglected in finding the last value of η are of the order η^3, or b^{-6}; the value of η may therefore be regarded as accurate as far as terms in b^{-4}, so that (30) gives the solution of (29) as far as terms in b^{-2}.

In the accompanying table, Columns 2 and 3 show the approximations to n, for different values of b, given by equations (24) and (28) respectively. Equation (27) would give values less by ·122, equation (25) values greater by 1, than those in the second column. Except for small values of b, the values from equations (28·1) and (30) coincide with those in the third column, and those from equation (26) are less than the latter by ·901. In the fourth column are given the values of n that we set out to find, in the remaining columns the values of ${}_nP_b$ provided by equations (16), (13), and (19) respectively.

b	Approximations to n		n	Approximations to ${}_nP_b$		
	(i) by (24)	(ii) by (28)		(i) by (16)	(ii) by (13)	(iii) by (19)
1	1·653	1·114	1	—	—	—
2	8·247	7·702	6	—	—	—
3	19·238	18·692	17	—	*·4806824*	·4806481
4	34·625	34·079	34	—	*·4995598*	·4995520
5	54·408	53·863	53	—	*·4966174*	·4966140
6	78·587	78·041	78	—	*·4998968*	·4998953
7	107·162	106·617	105	·49859	·4967030	·4967534
8	140·134	139·588	138	·49896	·4975330	·4975636
9	177·502	176·956	175	·49873	·4976108	·4976290
10	219·266	218·720	218	·50018	·4992850	·4992970
20	878·699	878·153	878	·50018	·4999620	·4999627
30	1977·753	1977·207	1976	·49997	·4998690	·4998691
40	3516·429	3515·883	3514	·49994	·4998850	·4998851
50	5494·727	5494·182	5494	·50003	·4999929	·4999928
60	7912·647	7912·101	7912	·50002	·4999971	·4999972
70	10770·189	10769·643	10768	·49995	·4999672	·4999673
80	14067·353	14066·807	14066	·49997	·4999876	·4999877
90	17804·138	17803·593	17802	·499980	·4999809	·4999808
100	21980·545	21980·000	21980	·500008	·4999999	·49999996

Until $n+b$ becomes greater than 100, we can find the true value of ${}_nP_b$ from tables of the Incomplete Beta-Function; these true values are shown in italics in Column 6 for the values 3, 4, 5, and 6 of b. It will be seen that even for such small values of b (19) provides a very close approximation, and we may expect it to improve as b increases. The approximations to ${}_nP_b$ provided by (16) and (13) diverge somewhat widely from the true values when b is very small, but the close agreement between the lower portions of Columns 6 and 7 indicates that the error in (13) very quickly disappears.

Some interest attaches to the value of n corresponding to $b=100$. It is for this value of b that Laplace says: "Il y a donc alors du désavantage à parier un contre un que A gagnera la partie dans 23780 coups; mais il y a de l'avantage à parier qu'il la gagnera dans 23781 coups." The difference between Laplace's result and our own is due in part to the fact that he was not able to refer to the tables that are nowadays available; but in any case Laplace would appear to have lost sight of the approximate nature of his result, since the odds are exactly the same that the set will be ended in 23780 games, as in 23781. Actually, we find from equation (19),

$$_{23780}P_{100} = \cdot 516687.$$

§ 5. Historical Note.

The problem of the Duration of Play is one of the oldest in the Calculus of Probabilities*, and several of the results given above are by no means new. Perhaps the best known of them are the expressions for B's chance of ultimately being ruined, namely

$$P_b = \frac{1-\left(\frac{q}{p}\right)^a}{1-\left(\frac{q}{p}\right)^{a+b}}, \qquad (p \neq q) \tag{11·1}$$

$$P_b = \frac{a}{a+b}, \qquad (p=q) \tag{11·2}$$

$$\begin{cases} P_b = \left(\frac{p}{q}\right)^b, & (a=\infty,\ q>p) \\ P_b = 1\ , & (a=\infty,\ q\leqslant p) \end{cases} \tag{6}$$

and the expression for B's chance of being ruined in a given number of games, when a is infinite, in the form

$$_{b+2i+1}P_b = {}_{b+2i}P_b = p^b\left[1 + b.pq + \frac{b(b+3)}{2}p^2q^2 + \ldots + \frac{b(b+2i-1)!}{i!(b+i)!}p^iq^i\right]. \tag{3}$$

* I have to thank Professor Pearson for pointing out that it was in Huygens' small tract, *Van Rekeningh in Spelen van Geluck*, in the course of which he considers the somewhat similar Problem of Points, that the Calculus of Probabilities originated. Huygens' problem is this: Several players engage in a set, he that first gains a certain number of games being the winner; given the number of games still required by the various players, determine their chances of winning. Huygens communicated his tract to his teacher, Franciscus van Schooten, who published a Latin translation of it in 1657 as an appendix to his *Exercitationum Mathematicarum Libri Quinque*; the vernacular version appeared in 1660, and English translations were published in 1692 or so by Dr Arbuthnot, and in 1714 by W. Browne. It is true that Pascal and Fermat were discussing questions of chance in their correspondence three or four years before Huygens' tract appeared, and Huygens himself says in his preface, "Sciendum vero, quod jam pridem inter praestantissimos tota Gallia Geometras calculus hic agitatus fuerit, ne quis indebitam mihi primae inventionis gloriam hac in re tribuat." But the Pascal-Fermat letters remained unpublished for another twenty years, and it was Huygens' tract that inspired the work of Montmort and de Moivre. In his preface to *The Doctrine of Chances*, de Moivre explicitly states that when he wrote his "Specimen" he "had not at that time read anything concerning this Subject, but Mr Huygens' Book *de Ratiociniis in Ludo Aleae*, and a little English piece which was properly a Translation of it."

This last result was first given by de Moivre, in his discussion of Problem LXV in *The Doctrine of Chances* (3rd edition, 1756). The equivalent result ((3·12) and (3·22)), and the expression (9·1) for the probability when a is not infinite that B will lose his last counter on the $(b+2i)$th game, were also given by de Moivre (*loc. cit.* Problems LXV and LXIV), although they were developed for the present paper before his work on the subject was consulted. De Moivre's solutions to Problems LXIV and LXV are given without any demonstration, but his solutions to Problems LVIII to LXIII, which also deal with the Duration of Play, leave little doubt that he reached his results by the inductive method indicated above (§ 2). Lagrange (see below) supplied proofs for equations (3), (3·12), and (3·22), but we have not been able to find any previous demonstration of (9·1).

Lagrange discusses the problem in the latter part of his "Recherches sur les suites récurrentes..., ou sur l'intégration des équations linéaires aux différences finies et partielles; et sur l'usage de ces équations dans la théorie des hazards" (*Nouveaux Mémoires de l'Académie Royale*, Berlin, 1775*). In his solution to Problème V (pp. 253—256, and pp. 258—261), Lagrange obtains de Moivre's two values for ${}_{b+2i}P_b$ in the case of a infinite; in Problème VI he finds for the general case the probability that either A or B will be broken in a given number of games, and indicates, without actually arriving at, a solution "qui répond à la méthode du Prob. LXIII...de Moivre" (pp. 261—265). Lagrange's solutions undoubtedly have, as he claims, the advantage of being "plus analytiques" than de Moivre's, but whether they are also "plus directes" seems a more open question.

Following Lagrange, Laplace demonstrated our equation (3) by means of his generating functions (*Théorie Analytique des Probabilités*, 1847 edition, p. 256) then Ampère, saying "j'ai banni de ces démonstrations les méthodes d'induction, dont on fait, à ce qu'il semble, trop d'usage dans la théorie des probabilités," established equation (2·31), and thence equation (3), by purely algebraic considerations of the possible combinations of gains and losses (*Considérations sur la Théorie Mathématique du Jeu*, Lyon, 1802). The probabilities (equations (6), (11·1), and (11·2)) that B will ultimately be ruined were first given by Ampère in this same memoir, but it is worth noticing that they follow immediately from one of Laplace's results (*loc. cit.* p. 254, equation H). He finds

$$
{}_{b+2i}P_b = \frac{p^b(p^a-q^a)}{p^{a+b}-q^{a+b}} - \frac{2^{b+2i+2}\,p^b\,(pq)^{i+1}}{a+b}
$$

$$
\times \sum_{r=0}^{\left[\frac{a+b-3}{2}\right]} \left\{ \frac{\sin\dfrac{2(r+1)\pi}{a+b}\,\sin\dfrac{(r+1)b\pi}{a+b}\left(\cos\dfrac{(r+1)\pi}{a+b}\right)^{b+2i}}{p^2-2pq\cos\dfrac{2(r+1)\pi}{a+b}+q^2} \right\} \dagger.
$$

* Lagrange's paper was read in 1776 and published in 1777.

† Laplace gives the upper limit of r in the sum, when $(a+b)$ is odd, as $\frac{1}{2}(a+b-1)$, but this is really the upper limit of $(r+1)$.

For all non-vanishing terms in the sum,

$$4pq\cos^2\frac{(r+1)\pi}{a+b}<1;$$

accordingly, as $i\to\infty$ each term of the sum tends to zero, giving (11·1) and (11·2), and thence, when we make $a\to\infty$, (6).

What seems to me to be the most elegant discussion of the problem is contained in Robert Leslie Ellis's paper "On the Solution of Equations in Finite Differences" (*Cambridge Mathematical Journal*, No. XXII, Vol. IV, 1844; reprinted in *Mathematical and other Writings*, 1863, pp. 203—211). Ellis obtains equations (9·7) and (9·8), but restricts m in the first of these equations to the range $1<m<a+b-1$. (9·7) possesses the same symmetrical character as (9·3), from which we have derived it; Ellis remarks on this symmetry, adding "the result, however, which is the interpretation of this symmetry, may probably be obtained by general considerations."

From (9·8) and (11·1) we have

$${}_nP_b=\frac{1-\left(\frac{q}{p}\right)^a}{1-\left(\frac{q}{p}\right)^{a+b}}-\sum_{n+1}^{\infty}p_{0,n},$$

or

$$(31)\quad {}_nP_b=\frac{1-\left(\frac{q}{p}\right)^a}{1-\left(\frac{q}{p}\right)^{a+b}}-\frac{(4pq)^{\frac{n+1}{2}}}{a+b}\left(\frac{p}{q}\right)^{\frac{b}{2}}\sum_{r=1}^{a+b-1}\frac{\sin\frac{r\pi}{a+b}\sin\frac{rb\pi}{a+b}\left(\cos\frac{r\pi}{a+b}\right)^n}{1-2\sqrt{pq}\cos\frac{r\pi}{a+b}}.$$

The sum of the rth and the $(a+b-r)$th terms in the second member of this equation is

$$\frac{\sin\frac{r\pi}{a+b}\sin\frac{rb\pi}{a+b}\left(\cos\frac{r\pi}{a+b}\right)^n}{1-2\sqrt{pq}\cos\frac{r\pi}{a+b}}+\frac{\sin\left(\pi-\frac{r\pi}{a+b}\right)\sin\left(b\pi-\frac{rb\pi}{a+b}\right)\left\{\cos\left(\pi-\frac{r\pi}{a+b}\right)\right\}^n}{1-2\sqrt{pq}\cos\left(\pi-\frac{r\pi}{a+b}\right)}$$

$$=\frac{\sin\frac{r\pi}{a+b}\sin\frac{rb\pi}{a+b}\left(\cos\frac{r\pi}{a+b}\right)^n}{p^2-2pq\cos\frac{2r\pi}{a+b}+q^2}$$

$$\times\left\{1+2\sqrt{pq}\cos\frac{r\pi}{a+b}+(-1)^{n+b-1}\left(1-2\sqrt{pq}\cos\frac{r\pi}{a+b}\right)\right\}.$$

Thus (31) is equivalent to

$$(31{\cdot}1)\quad {}_nP_b=\frac{1-\left(\frac{q}{p}\right)^a}{1-\left(\frac{q}{p}\right)^{a+b}}-\frac{2\,(4pq)^{\frac{n'+1}{2}}}{a+b}\left(\frac{p}{q}\right)^{\frac{b}{2}}\sum_{r=1}^{\left[\frac{a+b-1}{2}\right]}\frac{\sin\frac{r\pi}{a+b}\sin\frac{rb\pi}{a+b}\left(\cos\frac{r\pi}{a+b}\right)^{n'}}{p^2-2pq\cos\frac{2r\pi}{a+b}+q^2},$$

where $n'=n+1$ or n, according as $(n+b)$ is even or odd.

If we write $(b+2i)$ for n and $(r+1)$ for r, (31·1) becomes Laplace's equation H.

Let ${}_nQ_a$ be the probability that A will lose his last counter on or before the nth game. From (31) we have, by interchanging a with b, and p with q,

$${}_nQ_a = \frac{1-\left(\frac{p}{q}\right)^b}{1-\left(\frac{p}{q}\right)^{a+b}} - \frac{(4pq)^{\frac{n+1}{2}}}{a+b}\left(\frac{q}{p}\right)^{\frac{a}{2}} \sum_{r=1}^{a+b-1} \frac{\sin\frac{r\pi}{a+b}\sin\frac{ra\pi}{a+b}\left(\cos\frac{r\pi}{a+b}\right)^n}{1-2\sqrt{pq}\cos\frac{r\pi}{a+b}};$$

this equation, with (31), gives, for the probability that the set will end before the $(n+1)$th game,

$$(32)\quad {}_nP_b + {}_nQ_a = 1 - \frac{(4pq)^{\frac{n+1}{2}}}{a+b}\left(\frac{p}{q}\right)^{\frac{b}{2}} \times \sum_{r=1}^{a+b-1} \frac{\sin\frac{r\pi}{a+b}\left(\cos\frac{r\pi}{a+b}\right)^n}{1-2\sqrt{pq}\cos\frac{r\pi}{a+b}} \left\{\sin\frac{rb\pi}{a+b} + \left(\frac{q}{p}\right)^{\frac{a+b}{2}}\sin\frac{ra\pi}{a+b}\right\}.$$

Since $\sin\frac{ra\pi}{a+b} = (-)^{r-1}\sin\frac{rb\pi}{a+b}$, (32) reduces, when the necessary changes in notation are made, to Lagrange's second solution to his Problème VI (*loc. cit.* pp. 265—269).

If we make $b=a$, we get, from (31·1),

$$(32\cdot1)\quad {}_nP_a + {}_nQ_a = 1 - \left\{\left(\frac{p}{q}\right)^{\frac{a}{2}} + \left(\frac{q}{p}\right)^{\frac{a}{2}}\right\} \frac{(4pq)^{\frac{n'+1}{2}}}{2a} \sum_{r=1}^{a-1} \frac{\sin\frac{r\pi}{a}\sin\frac{r\pi}{2}\left(\cos\frac{r\pi}{2a}\right)^{n'-1}}{p^2-2pq\cos\frac{r\pi}{a}+q^2},$$

so that the probability that the set will not end in n games is

$$(32\cdot2)\quad 1-{}_nP_a-{}_nQ_a = \frac{p^a+q^a}{(pq)^{\frac{a}{2}-1}}\frac{(4pq)^{\frac{n'-1}{2}}}{a} \sum_{s=0}^{\left[\frac{a-2}{2}\right]} \frac{(-1)^s\sin\frac{2s+1}{a}\pi\left(\cos\frac{2s+1}{2a}\pi\right)^{n'-1}}{p^2-2pq\cos\frac{2s+1}{a}\pi+q^2};$$

if we now make $p=q=\frac{1}{2}$, we get

$$(32\cdot3)\qquad 1-{}_nP_a-{}_nQ_a = \frac{1}{a}\sum_{s=0}^{\left[\frac{a-2}{2}\right]} \frac{(-1)^s\left(\cos\frac{2s+1}{2a}\pi\right)^{n'}}{\sin\frac{2s+1}{2a}\pi}.$$

De Moivre gives the last of these results (*loc. cit.* Problem LXVIII), but instead of (32·2) he gives, if we do not mistake him,

$$1-{}_nP_a-{}_nQ_a = \frac{p^n+q^n}{(pq)^{\frac{a}{2}-1}}\frac{(4pq)^{\frac{n}{2}}}{a} \sum_{s=0}^{\left[\frac{a-2}{2}\right]} \frac{(-1)^s\sin\frac{2s+1}{a}\pi\left(\cos\frac{2s+1}{2a}\pi\right)^{n}}{p^2-2pq\cos\frac{2s+1}{a}\pi+q^2}$$

(*loc. cit.* Problem LXIX).

Except for those mentioned above, I believe the results of this paper to be new; in the case of the others, the methods by which I have established them seem to me to be simpler than those previously used, and to link together results that until now have appeared somewhat disconnected. The use of the Incomplete Beta-Function, in particular, renders almost intuitive the transition from B's chance of losing all his possessions in a given number of games, to his chance of doing so ultimately, a transition previously effected only in the case $a = \infty$, and by Ampère's very elaborate algebra.

This paper originated from some remarks in Professor Pearson's lectures on Laplace; I wish to thank him for his suggestions and advice.

ON THE NATURE OF THE RELATIONSHIP BETWEEN TWO OF "STUDENT'S" VARIATES (z_1 AND z_2) WHEN SAMPLES ARE TAKEN FROM A BIVARIATE NORMAL POPULATION.

By KARL PEARSON, F.R.S.

(1) It is well known that "Student" first introduced the study of the variate

$$z = \frac{\text{Mean of Sample} - \text{Mean of Parent Population}}{\text{Standard Deviation of Sample}}$$

as a method of testing whether a small sample has been drawn from a parent population of which the mean has been ascertained*. The method has been developed by several writers since the appearance of "Student's" original memoir. The value of z will certainly determine whether it be exceedingly improbable that the sample was drawn from the supposed parent population, but in my opinion it does not justify us in asserting it probably has been, if we find the value of z has a high degree of probability. The numerator and denominator of z are—at any rate in the proof provided by "Student," i.e. selection from a normal parent population—independent variables. There is nothing to check their variations being in the same direction, and their ratio z may take a very probable value, although individually they might be highly improbable as selections from the given parent population. Further, if the mean of the parent population be so well known that it can be safely used in the numerator, then it would appear that the standard deviation can also be safely determined, and we have two variates instead of a single one to compare with those of the sample. Such a comparison has always seemed to me safer than arguing from a single ratio. But "Student" uses his formula to compare two samples from two populations. Let these variates be given by x and y with standard deviations and means σ_1, σ_2, $\bar{x}$ and $\bar{y}$ for the samples, and Σ_1, Σ_2, m_1, m_2 for the corresponding parent populations. x and y may be correlated with correlation coefficient in the parent population $= \rho$, and in the samples $= r$. Now if x and y both follow a normal distribution, so will the difference of their differences from their means. In other words the ratio will be

$$z = \frac{(\bar{x} - m_1) - (\bar{y} - m_2)}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2r\sigma_1\sigma_2}}.$$

If now we ask whether x and y are *independent* samples from the *same* population, then we may suppose $m_1 = m_2$ and $r = 0$ to get our result. If they are not independent samples, we may put $m_1 = m_2$ but are not justified in putting $r = 0$. The two cases

$$z = \frac{\bar{x} - \bar{y}}{\sqrt{\sigma_1^2 + \sigma_2^2}} \quad \text{and} \quad z' = \frac{\bar{x} - \bar{y}}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2r\sigma_1\sigma_2}}$$

* *Biometrika*, Vol. VI. pp. 7—8.

may lead us to very different conclusions. Both z and z' get rid of a possibly unknown m of the parent population, but the second does not really get rid of the unknown r by the simple process of finding the standard deviation of $x-y$. The wide range of the coefficient of correlation r in *small* samples from a population of correlation ρ is well known, and appears only to be screened in taking the standard deviation of the difference. It seems necessary therefore to be sure that our two samples are wholly independent before using z. If they turn out not to be, i.e. if m_1 is very improbably equal to m_2, then we certainly are not justified when dealing with correlated samples in using z' where m_1 is put equal to m_2.

We may illustrate this in the following manner by asking whether the older generation is of less stature than the succeeding generation. We take a sample of fathers and a sample of sons, not sons of those fathers, and find z is sufficiently small for it to be probable that $m_1 = m_2$. We now take the fathers and sons to be correlated individuals, and find owing to the term in r, that z' is so large that it is unreasonable to suppose that m_1 may be put equal to m_2 in the case of sons of the same fathers. It will I think be clear that z' cannot determine what will happen in the case of z. For example, if we test for the relative effectiveness of two drugs or two methods of factory production on the *same* groups of individuals and find a significant difference, we have not obtained evidence that there would be a significant difference had the same drugs or same methods of production been tested on different groups of individuals*.

Notwithstanding the need for caution in the use of z, and the undesirability of exaggerating the efficiency of z tests, it seemed to me worth while to inquire into the relationship of two variates measured by "Student's" ratios.

(2) *Correlation Surface of z_1 and z_2.*

Using the same notation as in the preceding section we take

$$z_1 = (\bar{x} - m_1)/\sigma_1, \quad z_2 = (\bar{y} - m_2)/\sigma_2,$$

and we suppose the normal parent population defined by m_1, m_2, Σ_1, Σ_2 and correlation ρ. For brevity we may write $s_1 = \Sigma_1\sqrt{1-\rho^2}/\sqrt{n}$, $s_2 = \Sigma_2\sqrt{1-\rho^2}/\sqrt{n}$, where n is the size of the sample; r will represent the correlation in a particular sample. Z will denote the ordinate of any frequency surface or curve and Z_0 a constant independent of the variates of the particular sample. The constants of the square brackets following a frequency curve denote the appropriate element of volume. The correlation surface for the five variables $\bar{x}$, $\bar{y}$, σ_1, σ_2 and r is

$$Z = Z_0 e^{-\frac{1}{2}\frac{n}{1-\rho^2}\left\{\frac{(\bar{x}-m_1)^2}{\Sigma_1^2} - \frac{2\rho(\bar{x}-m_1)(\bar{y}-m_2)}{\Sigma_1\Sigma_2} + \frac{(\bar{y}-m_2)^2}{\Sigma_2^2}\right\}}$$
$$\times e^{-\frac{1}{2}\frac{n}{1-\rho^2}\left\{\frac{\sigma_1^2}{\Sigma_1^2} - \frac{2r\rho\sigma_1\sigma_2}{\Sigma_1\Sigma_2} + \frac{\sigma_2^2}{\Sigma_2^2}\right\}} (\sigma_1\sigma_2)^{n-2}(1-r^2)^{\frac{n-4}{2}} [d\bar{x}d\bar{y}d\sigma_1 d\sigma_2 dr]$$
$$\ldots\ldots(\text{i}).$$

* For further illustration see *Biometrika*, Vol. XXII. pp. 263—270.

We will first integrate this expression with regard to r between the limits -1 and $+1$. The result is given as Equation (v) of *Biometrika*, Vol. XVII. p. 177, and substituting for $\bar{x}-m_1$ and $\bar{y}-m_2$, we have *

$$Z = Z_0' e^{-\frac{1}{2}\left(\frac{(1+z_1^2)\sigma_1^2}{s_1^2} - \frac{2\rho z_1 z_2 \sigma_1\sigma_2}{s_1 s_2} + \frac{(1+z_2^2)\sigma_2^2}{s_2^2}\right)} (\sigma_1\sigma_2)^{n-1}$$
$$\times \left(1 + \frac{\rho^2}{1!}\frac{\sigma_1^2\sigma_2^2}{(2n-2)s_1^2 s_2^2} + \frac{\rho^4}{2!}\frac{\sigma_1^4\sigma_2^4}{(2n-2)(2n+2)s_1^4 s_2^4} + \cdots\right.$$
$$\left. + \frac{\rho^{2p}}{p!}\frac{\sigma_1^{2p}\sigma_2^{2p}}{(2n-2)(2n+2)\ldots(2n+4p-6)s_1^{2p}s_2^{2p}} + \cdots\right)[dz_1 dz_2 d\sigma_1 d\sigma_2] \ldots(\text{ii}).$$

To obtain the surface of frequency of z_1, z_2 we need to integrate this for σ_1 and σ_2, from 0 to ∞ in both cases. It is necessary first to expand the exponential term $e^{\frac{\rho z_1 z_2}{s_1 s_2}\sigma_1\sigma_2}$, and we have

$$Z = Z_0' e^{-\frac{1}{2}\frac{1+z_1^2}{s_1^2}\sigma_1^2} e^{-\frac{1}{2}\frac{1+z_2^2}{s_2^2}\sigma_2^2} \times \sum_{p'=0}^{\infty}\frac{1}{p'!}\left(\rho z_1 z_2 \frac{\sigma_1\sigma_2}{s_1 s_2}\right)^{p'} \times \left(\frac{\sigma_1}{s_1}\frac{\sigma_2}{s_2}\right)^{n-1}(s_1 s_2)^n$$
$$\times \sum_{p=0}^{p=\infty}\frac{\rho^{2p}}{p!}\frac{\sigma_1^{2p}\sigma_2^{2p}}{s_1^{2p}s_2^{2p}}\frac{1}{(2n-2)(2n+2)\ldots(2n+4p-6)}\left[d\left(\frac{\sigma_1}{s_1}\right)d\left(\frac{\sigma_2}{s_2}\right)\right].$$

Take $v_1 = \frac{1}{2}\frac{1+z_1^2}{s_1^2}\sigma_1^2$, $v_2 = \frac{1}{2}\frac{1+z_2^2}{s_2^2}\sigma_2^2$, then $dv_1 = \sigma_1 d\sigma_1 \frac{1+z_1^2}{s_1^2} = \lambda_1 \frac{\sigma_1}{s_1}\frac{d\sigma_1}{s_1}$, say, and $dv_2 = \sigma_2 d\sigma_2 \frac{1+z_2^2}{s_2^2} = \lambda_2 \frac{\sigma_2}{s_2}\frac{d\sigma_2}{s_2}$, say,

$$Z = Z_0' (s_1 s_2)^n e^{-v_1 - v_2} \sum_{p'=0}^{\infty}\sum_{p=0}^{\infty} 2^{p'+2p+n-2}(z_1 z_2)^{p'}\frac{\rho^{p'+2p}}{p'!\,p!}$$
$$\times \frac{1}{(2n-2)(2n+2)\ldots(2n+4p-6)} \times \frac{v_1^{\frac{1}{2}(p'+2p+n-2)}\, v_2^{\frac{1}{2}(p'+2p+n-2)}}{\lambda_1^{\frac{1}{2}(p'+2p+n)}\,\lambda_2^{\frac{1}{2}(p'+2p+n)}}[dv_1 dv_2].$$

Integrate for v_1 and v_2 from 0 to ∞, and we have for the surface of frequency of z_1 and z_2

$$Z = Z_0' (s_1 s_2)^n \sum_{p'=0}^{\infty}\sum_{p=0}^{\infty} 2^{p'+2p+n-2}\frac{\rho^{p'+2p}(z_1 z_2)^{p'}}{p'!\,p!}$$
$$\times \frac{1}{(2n-2)(2n+2)\ldots(2n+4p-6)} \times \frac{\Gamma^2\left(\frac{1}{2}(p'+2p+n)\right)}{(\lambda_1\lambda_2)^{\frac{1}{2}(p'+2p+n)}}.$$

We may write this in a somewhat different form, namely:

$$Z = Z_0'' \frac{1}{(\lambda_1\lambda_2)^{\frac{1}{2}n}} \sum_{p'=0}^{\infty}\frac{2^{p'}\rho^{p'}}{p'!}\left(\frac{z_1 z_2}{\sqrt{\lambda_1\lambda_2}}\right)^{p'}\Gamma^2\left(\tfrac{1}{2}(p'+n)\right)$$
$$\times \left(1 + \frac{2^2\rho^2\left(\frac{1}{2}(p'+n)\right)^2}{1!\,\lambda_1\lambda_2(2n-2)} + \frac{2^4\rho^4\left(\frac{1}{2}(p'+n)\right)^2\left(\frac{1}{2}(p'+n)+1\right)^2}{2!(\lambda_1\lambda_2)^2(2n-2)(2n+2)} + \ldots\right)$$
$$= Z_0'' \frac{1}{(\lambda_1\lambda_2)^{\frac{1}{2}n}} \sum_{p'=0}^{\infty}\frac{2^{p'}\rho^{p'}}{p'!}\left(\frac{z_1 z_2}{\sqrt{\lambda_1\lambda_2}}\right)^{p'}\Gamma^2\left(\tfrac{1}{2}(p'+n)\right)$$
$$\times F\left(\tfrac{1}{2}(p'+n), \tfrac{1}{2}(p'+n), \tfrac{1}{2}(n-1), \frac{\rho^2}{\lambda_1\lambda_2}\right).$$

* $Z_0' = Z_0 B\left(\frac{1}{2}, \frac{1}{2}(n-2)\right)$. We shall however pay no attention to the changes in the constant Z_0.

But by Euler's transformation of the hypergeometrical function

$$F\left(\tfrac{1}{2}(p'+n), \tfrac{1}{2}(p'+n), \tfrac{1}{2}(n-1), \frac{\rho^2}{\lambda_1\lambda_2}\right) = \left(1-\frac{\rho^2}{\lambda_1\lambda_2}\right)^{-(\frac{1}{2}(n+1)+p')} \times F\left(-\tfrac{1}{2}(p'+1), -\tfrac{1}{2}(p'+1), \tfrac{1}{2}(n-1), \frac{\rho^2}{\lambda_1\lambda_2}\right),$$

and accordingly

$$Z = Z_0'' \frac{\sqrt{\lambda_1\lambda_2}}{(\lambda_1\lambda_2-\rho^2)^{\frac{1}{2}(n+1)}} \sum_{p'=0}^{\infty} \frac{\Gamma^2(\frac{1}{2}(p'+n))}{p'!} \left(\frac{2\rho z_1 z_2\sqrt{\lambda_1\lambda_2}}{\lambda_1\lambda_2-\rho^2}\right)^{p'} \times F\left(-\tfrac{1}{2}(p'+1), -\tfrac{1}{2}(p'+1), \tfrac{1}{2}(n-1), \frac{\rho^2}{\lambda_1\lambda_2}\right),$$

or, substituting for the λ's,

$$Z = Z_0'' \sqrt{\frac{(1+z_1^2)(1+z_2^2)}{((1+z_1^2)(1+z_2^2)-\rho^2)^{n+1}}} \sum_{p'=0}^{\infty} \frac{\Gamma^2(\frac{1}{2}(p'+n))}{p'!} \left(\frac{2\rho z_1 z_2\sqrt{(1+z_1^2)(1+z_2^2)}}{(1+z_1^2)(1+z_2^2)-\rho^2}\right)^{p'} \times F\left(-\tfrac{1}{2}(p'+1), -\tfrac{1}{2}(p'+1), \tfrac{1}{2}(n-1), \frac{\rho^2}{(1+z_1^2)(1+z_2^2)}\right) \quad \text{...(iii).}$$

When $\rho = 0$, this reduces, as it should do, to

$$Z = Z_0'' \frac{1}{(1+z_1^2)^{\frac{1}{2}n}} \times \frac{1}{(1+z_2^2)^{\frac{1}{2}n}},$$

for the case of z_1 and z_2 independent.

I have not succeeded in reducing (iii) in the general case to any more concise form, and it appears too complicated for any numerical reduction for particular values of n and ρ. I leave it in the hopes that a stronger algebraist may possibly achieve something with it. We need not, however, despair of learning something about the nature of the z_1, z_2 frequency surface, for we can calculate its principal characters. I shall now proceed to determine (*a*) the coefficient of correlation of z_1 and z_2, (*b*) the regression curve of z_1 on z_2, and (*c*) the scedasticity of the arrays of z_1 for a given z_2.

(3) *Determination of the Correlation between z_1 and z_2.*

We have seen that the frequency surface of r, σ_1, σ_2, $\bar{x}$, $\bar{y}$ is given by

$$Z = Z_0 e^{-\frac{1}{2}\frac{n}{1-\rho^2}\left\{\frac{(\bar{x}-m_1)^2}{\Sigma_1^2} - \frac{2\rho(\bar{x}-m_1)(\bar{y}-m_2)}{\Sigma_1\Sigma_2} + \frac{(\bar{y}-m_2)^2}{\Sigma_2^2}\right\}} \times e^{-\frac{1}{2}\frac{n}{1-\rho^2}\left\{\frac{\sigma_1^2}{\Sigma_1^2} - 2r\rho\frac{\sigma_1\sigma_2}{\Sigma_1\Sigma_2} + \frac{\sigma_2^2}{\Sigma_2^2}\right\}} (\sigma_1\sigma_2)^{n-2}(1-r^2)^{\frac{n-4}{2}},$$

with the element dV of "volume" $= d\bar{x}d\bar{y}d\sigma_1 d\sigma_2 dr$. Now to obtain the correlation of z_1 and z_2 we need, if N = number of samples,

$$P_{z_1 z_2} = \frac{1}{N}\int Z\left(\frac{\bar{x}-m_1}{\sigma_1}\right)\left(\frac{\bar{y}-m_2}{\sigma_2}\right) dV,$$

or the product moment of z_1, z_2, the integral being taken over the whole of space. But

$$P_{z_1z_2}=\frac{Z_0}{N}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{-\frac{1}{2}\frac{n}{1-\rho^2}\left\{\frac{(\bar{x}-m_1)^2}{\Sigma_1^2}-\frac{2\rho(\bar{x}-m_1)(\bar{y}-m_2)}{\Sigma_1\Sigma_2}+\frac{(\bar{y}-m_2)^2}{\Sigma_2^2}\right\}}\times d\bar{x}\,d\bar{y}\,(\bar{x}-m_1)(\bar{y}-m_2)$$
$$\times\int_0^\infty\int_0^\infty\int_{-1}^{+1}e^{-\frac{1}{2}\frac{n}{1-\rho^2}\left\{\frac{\sigma_1^2}{\Sigma_1^2}-2r\rho\frac{\sigma_1\sigma_2}{\Sigma_1\Sigma_2}+\frac{\sigma_2^2}{\Sigma_2^2}\right\}}(\sigma_1\sigma_2)^{n-3}(1-r^2)^{\frac{n-4}{2}}\,d\sigma_1d\sigma_2dr.$$

Now the integral with regard to $d\bar{x}\,d\bar{y}$ can be taken at once. It is

$$2\pi\sqrt{1-\rho^2}\frac{\Sigma_1\Sigma_2}{n}\times\rho\frac{\Sigma_1\Sigma_2}{n}.$$

Hence

$$P_{z_1z_2}=\frac{Z_0}{N}2\pi\sqrt{1-\rho^2}\,\rho\frac{\Sigma_1^2\Sigma_2^2}{n^2}$$
$$\times\int_0^\infty\int_0^\infty\int_{-1}^{+1}e^{-\frac{1}{2}\frac{n}{1-\rho^2}\left\{\frac{\sigma_1^2}{\Sigma_1^2}-2r\rho\frac{\sigma_1\sigma_2}{\Sigma_1\Sigma_2}+\frac{\sigma_2^2}{\Sigma_2^2}\right\}}(\sigma_1\sigma_2)^{n-3}(1-r^2)^{\frac{n-4}{2}}\,d\sigma_1d\sigma_2dr.$$

But if $\gamma=\frac{n\rho}{1-\rho^2}\frac{\sigma_1\sigma_2}{\Sigma_1\Sigma_2}$, then the $\int_{-1}^{+1}e^{\gamma r}(1-r^2)^{\frac{n-4}{2}}\,dr$ is known to be*

$$B\left(\tfrac{1}{2},\frac{n-2}{2}\right)\left(1+\frac{\gamma^2}{1!}\frac{1}{2n-2}+\frac{\gamma^4}{2!}\frac{1}{(2n-2)(2n+2)}+\cdots\right.$$
$$\left.+\frac{\gamma^{2p}}{p!}\frac{1}{(2n-2)(2n+2)\ldots(2n+4p-6)}+\cdots\right).$$

Hence, writing as before, $s_1=\sqrt{1-\rho^2}\Sigma_1/\sqrt{n}$, $s_2=\sqrt{1-\rho^2}\Sigma_2/\sqrt{n}$, the second integral, I_2, in the value of $P_{z_1z_2}$ reduces to

$$B\left(\tfrac{1}{2},\frac{n-2}{2}\right)\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\left(\frac{\sigma_1^2}{s_1^2}+\frac{\sigma_2^2}{s_2^2}\right)}\left(\frac{\sigma_1\sigma_2}{s_1s_2}\right)^{n-3}(s_1s_2)^{n-2}$$
$$\times\left(1+\frac{\rho^2}{1!}\left(\frac{\sigma_1\sigma_2}{s_1s_2}\right)^2\frac{1}{2n-2}+\frac{\rho^4}{2!}\left(\frac{\sigma_1\sigma_2}{s_1s_2}\right)^4\frac{1}{(2n-2)(2n+2)}+\cdots\right.$$
$$\left.+\frac{\rho^{2p}}{p!}\left(\frac{\sigma_1\sigma_2}{s_1s_2}\right)^{2p}\frac{1}{(2n-2)(2n+2)\ldots(2n+4p-6)}+\ldots\right)\frac{d\sigma_1}{s_1}\frac{d\sigma_2}{s_2}.$$

Now put $\quad\frac{1}{2}\frac{\sigma_1^2}{s_1^2}=v_1,\quad\frac{1}{2}\frac{\sigma_2^2}{s_2^2}=v_2,$

and we have

$$I_2=B\left(\tfrac{1}{2},\frac{n-2}{2}\right)(s_1s_2)^{n-2}\int_0^\infty\int_0^\infty 2^{n-4}e^{-v_1-v_2}(v_1v_2)^{\frac{n-4}{2}}$$
$$\times\left(1+\frac{(2\rho)^2}{1!}\frac{v_1v_2}{2n-2}+\frac{(2\rho)^4}{2!}\frac{(v_1v_2)^2}{(2n-2)(2n+2)}+\cdots\right.$$
$$\left.+\frac{(2\rho)^{2p}}{p!}(v_1v_2)^p\frac{1}{(2n-2)(2n+2)\ldots(2n+4p-6)}+\ldots\right)dv_1dv_2$$

* *Biometrika*, Vol. XVII. p. 177.

$$= 2^{n-4} B\left(\tfrac{1}{2}, \frac{n-2}{2}\right)(s_1 s_2)^{n-2}\left(\Gamma^2\left(\frac{n-2}{2}\right) + \frac{(2\rho)^2}{1!}\Gamma^2\left(\frac{n}{2}\right)\frac{1}{2n-2}\right.$$
$$+ \frac{(2\rho)^4}{2!}\Gamma^2\left(\frac{n+2}{2}\right)\frac{1}{(2n-2)(2n+2)} + \dots$$
$$\left. + \frac{(2\rho)^{2p}}{p!}\frac{\Gamma^2\left(\frac{n-2+2p}{2}\right)}{(2n-2)(2n+2)\dots(2n+4p-6)} + \dots\right)$$
$$= 2^{n-4} B\left(\tfrac{1}{2}, \frac{n-2}{2}\right)(s_1 s_2)^{n-2}\Gamma^2\left(\frac{n-2}{2}\right)$$
$$\times\left(1 + \frac{(2\rho)^2}{1!}\left(\frac{n-2}{2}\right)^2\frac{1}{2n-2} + \frac{(2\rho)^4}{2!}\left(\frac{n-2}{2}\right)^2\left(\frac{n}{2}\right)^2\frac{1}{(2n-2)(2n+2)}\right.$$
$$\left. + \left(\frac{n-2}{2}\right)^2\left(\frac{n}{2}\right)^2\left(\frac{n+1}{2}\right)^2\frac{(2\rho)^6}{3!}\frac{1}{(2n-2)(2n+2)(2n+6)} + \dots\right)$$
$$= 2^{n-4} B\left(\tfrac{1}{2}, \frac{n-2}{2}\right)(s_1 s_2)^{n-2}\Gamma^2\left(\frac{n-2}{2}\right) F\left(\frac{n-2}{2}, \frac{n-2}{2}, \frac{n-1}{2}, \rho^2\right).$$

But by Euler's Theorem

$$F(\alpha, \beta, \gamma, x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma, \rho^2).$$

Accordingly

$$I_2 = 2^{n-4} B\left(\tfrac{1}{2}, \frac{n-2}{2}\right)(s_1 s_2)^{n-2}\Gamma^2\left(\frac{n-2}{2}\right)(1-\rho^2)^{-\frac{n-3}{2}} F\left(\tfrac{1}{2}, \tfrac{1}{2}, \frac{n-1}{2}, \rho^2\right).$$

Thus we have

$$P_{z_1 z_2} = \frac{Z_0}{N}\frac{2\pi}{(1-\rho^2)^{\frac{n}{2}}}\rho(s_1 s_2)^n 2^{n-4} B\left(\tfrac{1}{2}, \frac{n-2}{2}\right)\Gamma^2\left(\frac{n-2}{2}\right)$$
$$\times (1-\rho^2)^{-\frac{n-3}{2}} F\left(\tfrac{1}{2}, \tfrac{1}{2}, \frac{n-1}{2}, \rho^2\right) \dots\text{(iv)}.$$

We must now find Z_0 from the relation

$$N = \int z\, dV$$
$$= Z_0 2\pi\sqrt{1-\rho^2}\,\frac{\Sigma_1\Sigma_2}{n} \times \int_0^\infty\int_0^\infty\int_{-1}^{+1} (\sigma_1\sigma_2)^{n-2}(1-r^2)^{\frac{n-4}{2}}$$
$$\times e^{-\frac{1}{2}\frac{n}{1-\rho^2}\left(\frac{\sigma_1^2}{\Sigma_1^2} - \frac{2r\rho\,\sigma_1\sigma_2}{\Sigma_1\Sigma_2} + \frac{\sigma_2^2}{\Sigma_2^2}\right)} d\sigma_1 d\sigma_2 dr.$$

The integration with regard to r gives the same result as before, and the sole difference is the term $(\sigma_1\sigma_2)^{n-2}$ instead of $(\sigma_1\sigma_2)^{n-3}$. Accordingly

$$N = Z_0 2\pi\sqrt{1-\rho^2}\,\frac{\Sigma_1\Sigma_2}{n}(s_1 s_2)^{n-1} B\left(\tfrac{1}{2}, \frac{n-2}{2}\right)$$
$$\times\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\left(\frac{\sigma_1^2}{s_1^2} + \frac{\sigma_2^2}{s_2^2}\right)}\left(\frac{\sigma_1\sigma_2}{s_1 s_2}\right)^{n-3}\left(1 + \frac{\rho^2}{1!}\left(\frac{\sigma_1\sigma_2}{s_1 s_2}\right)^2\frac{1}{2n-2}\right.$$
$$\left. + \frac{\rho^4}{2!}\left(\frac{\sigma_1\sigma_2}{s_1 s_2}\right)^4\frac{1}{(2n-2)(2n+2)} + \dots\right)\frac{\sigma_1}{s_1} d\left(\frac{\sigma_1}{s_1}\right)\frac{\sigma_2}{s_2} d\left(\frac{\sigma_2}{s_2}\right).$$

Or, making the same transformation as before,

$$N = Z_0 2\pi \sqrt{1-\rho^2}\,\frac{\Sigma_1\Sigma_2}{n}(s_1s_2)^{n-1}B\left(\tfrac{1}{2},\frac{n-2}{2}\right)2^{n-3}$$
$$\times\int_0^\infty\int_0^\infty e^{-v_1-v_2}(v_1v_2)^{\frac{n-3}{2}}\left(1+\frac{(2\rho)^2}{1!}\frac{v_1v_2}{2n-2}+\frac{(2\rho)^4}{2!}\frac{(v_1v_2)^2}{(2n-2)(2n+2)}+\dots\right.$$
$$\left.+\frac{(2\rho)^{2p}}{p!}(v_1v_2)^p\frac{1}{(2n-2)(2n+2)\dots(2n+4p-6)}+\dots\right)dv_1dv_2$$
$$= Z_0\frac{2\pi}{\sqrt{1-\rho^2}}(s_1s_2)^nB\left(\tfrac{1}{2},\frac{n-2}{2}\right)2^{n-3}$$
$$\times\left(\Gamma^2\left(\frac{n-1}{2}\right)+\frac{(2\rho)^2}{1!}\frac{\Gamma^2\left(\frac{n+1}{2}\right)}{2n-2}+\frac{(2\rho)^4}{2!}\frac{\Gamma^2\left(\frac{n+3}{2}\right)}{(2n-2)(2n+2)}\right.$$
$$\left.+\frac{(2\rho)^6}{3!}\frac{\Gamma^2\left(\frac{n+5}{2}\right)}{(2n-2)(2n+2)(2n+6)}+\dots\right)$$
$$= Z_0\frac{2\pi}{\sqrt{1-\rho^2}}(s_1s_2)^nB\left(\tfrac{1}{2},\frac{n-2}{2}\right)2^{n-3}\Gamma^2\left(\frac{n-1}{2}\right)$$
$$\times\left(1+\frac{n-1}{2}\frac{\rho^2}{1!}+\frac{n-1}{2}\frac{n+1}{2}\frac{\rho^4}{2!}+\frac{n-1}{2}\frac{n+1}{2}\frac{n+3}{2}\frac{\rho^6}{3!}+\dots\right).$$

Thus
$$N=\frac{Z_0 2\pi(s_1s_2)^nB\left(\frac{1}{2},\frac{n-2}{2}\right)2^{n-3}\Gamma^2\left(\frac{n-1}{2}\right)}{(1-\rho^2)^{n/2}}.$$

Or
$$Z_0=\frac{N(1-\rho^2)^{n/2}}{2\pi(s_1s_2)^nB\left(\frac{1}{2},\frac{n-2}{2}\right)2^{n-3}\Gamma^2\left(\frac{n-1}{2}\right)}\quad\dots\dots\dots\dots\dots\dots\text{(v)}.$$

Returning to Equation (iv) and substituting we have

$$P_{z_1z_2}=\tfrac{1}{2}\rho F\left(\tfrac{1}{2},\tfrac{1}{2},\frac{n-1}{2},\rho^2\right)\Gamma^2\left(\frac{n-2}{2}\right)\Big/\Gamma^2\left(\frac{n-1}{2}\right).$$

But we need to divide by $\sigma_{z_1}\times\sigma_{z_2}$ to find $r_{z_1z_2}$. Now *

$$\sigma_{z_1}=\sigma_{z_2}=\frac{1}{\sqrt{n-3}}.$$

Accordingly we have

$$r_{z_1z_2}=\frac{n-3}{2}\frac{\Gamma^2\left(\frac{n-2}{2}\right)}{\Gamma^2\left(\frac{n-1}{2}\right)}\rho F\left(\tfrac{1}{2},\tfrac{1}{2},\frac{n-1}{2},\rho^2\right)$$
$$=\frac{\Gamma^2\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-3}{2}\right)}\rho F\left(\tfrac{1}{2},\tfrac{1}{2},\frac{n-1}{2},\rho^2\right)\dots\dots\dots\dots\dots\text{(v)}.$$

* See *Biometrika*, Vol. VI. p. 12.

TABLE I.

Correlation of z_1 and z_2 for various Values of ρ and n.

Samples of n.

$n=$	4	5	6	7	8	9	10	11	12
$\rho=0\cdot0$	·0000	·0000	·0000	·0000	·0000	·0000	·0000	·0000	·0000
0·1	·0638	·0786	·0850	·0884	·0906	·0921	·0932	·0940	·0946
0·2	·1282	·1579	·1705	·1773	·1816	·1845	·1867	·1883	·1896
0·3	·1940	·2384	·2570	·2671	·2734	·2777	·2808	·2832	·2850
0·4	·2620	·3209	·3453	·3584	·3665	·3720	·3759	·3789	·3813
0·5	·3333	·4063	·4360	·4517	·4614	·4678	·4725	·4760	·4787
0·6	·4097	·4960	·5307	·5480	·5587	·5658	·5709	·5747	·5776
0·7	·4936	·5919	·6293	·6482	·6594	·6667	·6718	·6756	·6785
0·8	·5933	·6976	·7357	·7541	·7646	·7713	·7760	·7793	·7819
0·9	·7129	·8204	·8540	·8687	·8766	·8814	·8847	·8869	·8887
1·0	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000

$n=$	13	14	15	16	17	18	19	20	21
$\rho=0\cdot0$	·0000	·0000	·0000	·0000	·0000	·0000	·0000	·0000	·0000
0·1	·0952	·0956	·0960	·0963	·0965	·0968	·0970	·0971	·0973
0·2	·1906	·1914	·1921	·1927	·1932	·1937	·1941	·1944	·1947
0·3	·2865	·2877	·2887	·2896	·2903	[illegible]	·2915	·2920	·2924
0·4	·3831	·3847	·3859	·3870	·3879	·3888	·3895	·3901	·3906
0·5	·4808	·4826	·4841	·4853	·4864	·4873	·4881	·4888	·4894
0·6	·5799	·5818	·5834	·5847	·5858	·5868	·5876	·5884	·5890
0·7	·6807	·6826	·6841	·6854	·6865	·6875	·6883	·6890	·6896
0·8	·7839	·7855	·7868	·7879	·7888	·7896	·7903	·7909	·7915
0·9	·8900	·8910	·8919	·8926	·8932	·8937	·8941	·8945	·8948
1·0	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000

$n=$	22	23	24	25	26	27	28	29	30
$\rho=0\cdot0$	·0000	·0000	·0000	·0000	·0000	·0000	·0000	·0000	·0000
0·1	·0974	·0976	·0977	·0978	·0979	·0980	·0980	·0981	·0982
0·2	·1950	·1952	·1955	·1957	·1959	·1960	·1962	·1963	·1965
0·3	·2928	·2932	·2935	·2938	·2941	·2943	·2946	·2948	·2950
0·4	·3911	·3916	·3920	·3923	·3927	·3930	·3933	·3935	·3938
0·5	·4900	·4905	·4910	·4914	·4918	·4921	·4924	·4927	·4930
0·6	·5896	·5902	·5906	·5911	·5915	·5918	·5922	·5925	·5927
0·7	·6902	·6907	·6912	·6916	·6920	·6923	·6926	·6929	·6932
0·8	·7919	·7924	·7928	·7931	·7934	·7937	·7940	·7942	·7944
0·9	·8951	·8954	·8956	·8958	·8960	·8962	·8964	·8965	·8966
1·0	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000

$n=$	50	52	100	102	400	402	1000	∞
$\rho=0\cdot0$	·0000	·0000	·0000	·0000	·0000	·0000	·0000	·0000
0·1	·0990	·0990	·0995	·0995	·0999	·0999	·1000	·1000
0·2	·1980	·1980	·1990	·1990	·1998	·1998	·1999	·2000
0·3	·2971	·2972	·2986	·2986	·2997	·2997	·2999	·3000
0·4	·3964	·3966	·3983	·3983	·3996	·3996	·3998	·4000
0·5	·4960	·4962	·4981	·4981	·4995	·4995	·4998	·5000
0·6	·5959	·5960	·5980	·5980	·5995	·5995	·5998	·6000
0·7	·6961	·6963	·6981	·6982	·6995	·6996	·6998	·7000
0·8	·7969	·7970	·7985	·7985	·7996	·7996	·7999	·8000
0·9	·8981	·8982	·8991	·8991	·8998	·8998	·8999	·9000
1·0	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000

But by *Biometrika*, Vol. XI. p. 336, the *mean* value of a correlation coefficient in samples of n is given by

$$\bar{r}_n = \frac{\Gamma^2\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)} \rho F\left(\tfrac{1}{2}, \tfrac{1}{2}, \frac{n+1}{2}, \rho^2\right) \quad \text{.....(vi).}$$

Change n to $(n-2)$ and we have the result that: *The mean value of the correlation coefficient in samples of* $(n-2)$ *from a parent population of correlation* ρ *is equal to the correlation of* z_1 *and* z_2 *in samples of size* n *from the same parent population.*

This is a somewhat remarkable theorem; it enables us at once to provide values of $r_{z_1 z_2}$ from those already calculated for $\bar{r}_n$. These are given in Table I.

(4) *Determination of the Regression Equation.*

A knowledge of the correlation coefficient of z_1 with z_2 is, however, of small service, if, the regression being non-linear, we have not some measure of its approach to linearity. We will therefore find the true regression of z_1 on z_2.

Let $\bar{z}_1$ be the mean value of the array of z_1 for a given value z_2 of z_2. We have, if n_{z_2} denote the frequency of the array of z_1's for the given z_2,

$$n_{z_2} \times \bar{z}_1 = \int_{-\infty}^{+\infty}\int_0^\infty\int_0^\infty Z z_1 d\sigma_1 d\sigma_2 dz_1,$$

where Z is the ordinate of the frequency surface. Hence

$$n_{z_2} \times \bar{z}_1 = Z_0 \int_0^\infty d\sigma_1 \int_0^\infty d\sigma_2 \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(z_1\frac{\sigma_1}{s_1} - \rho z_2 \frac{\sigma_2}{s_2}\right)^2 - \frac{1}{2}\frac{\sigma_2^2}{s_2^2}(1+z_2^2(1-\rho^2))}$$

$$\times e^{-\frac{1}{2}\frac{\sigma_1^2}{s_1^2}} \frac{s_1}{\sigma_1}\left(\left\{z_1\frac{\sigma_1}{s_1} - \rho z_2\frac{\sigma_2}{s_2}\right\} + \rho z_2 \frac{\sigma_2}{s_2}\right)(\sigma_1\sigma_2)^{n-1} Q\, dz,$$

where Q is the series

$$1 + \frac{\rho^2\sigma_1^2\sigma_2^2}{1!(2n-2)s_1^2 s_2^2} + \frac{\rho^4\sigma_1^4\sigma_2^4}{2!(2n-2)(2n+2)s_1^4 s_2^4} + \dots$$

$$+ \frac{\rho^{2p}\sigma_1^{2p}\sigma_2^{2p}}{p!(2n-2)(2n+2)\dots(2n+4p-6)s_1^{2p}s_2^{2p}} + \dots.$$

Writing $\kappa^2 = 1 + z_2^2(1-\rho^2)$ and $z_1\frac{\sigma_1}{s_1} - \rho z_2\frac{\sigma_2}{s_2} = u$, we have

$$n_{z_2} \times \bar{z}_1 = Z_0 \int_0^\infty d\sigma_1 \int_0^\infty d\sigma_2 \int_{-\infty}^{+\infty} e^{-\frac{1}{2}u^2}\frac{s_1^2}{\sigma_1^2}\left(u + \rho\frac{z_2\sigma_2}{s_2}\right) du\,(\sigma_1\sigma_2)^{n-1} Q e^{-\frac{1}{2}\left(\frac{\sigma_1^2}{s_1^2} + \frac{\sigma_2^2\kappa^2}{s_2^2}\right)}.$$

But $\int_{-\infty}^{+\infty} e^{-\frac{u^2}{2}} u\,du = 0$ and $\int_{-\infty}^{+\infty} e^{-\frac{1}{2}u^2} du = \sqrt{2\pi}$. Thus

$$n_{z_2} \times \bar{z}_1 = Z_0 \int_0^\infty d\sigma_1 \int_0^\infty d\sigma_2 \sqrt{2\pi}\,\rho z_2 \frac{s_1^2}{\sigma_1^2}\frac{\sigma_2}{s_2}(\sigma_1\sigma_2)^{n-1} Q e^{-\frac{1}{2}\frac{\sigma_1^2}{s_1^2}} e^{-\frac{1}{2}\frac{\sigma_2^2\kappa^2}{s_2^2}}.$$

Now put $\frac{1}{2}\frac{\sigma_1^2}{s_1^2} = v_1$ and $\frac{1}{2}\frac{\sigma_2^2\kappa^2}{s_2^2} = v_2$ and we find

$$n_{z_2} \times \tilde{z}_1 = Z_0\sqrt{2\pi}\frac{s_1^n s_2^n}{\kappa^{n+1}}\,2^{n-\frac{5}{2}}\rho z_2 \int_0^\infty\int_0^\infty e^{-v_1}e^{-v_2} \times \sum_{p=0}^{p=\infty}\left(\frac{2\rho}{\kappa}\right)^{2p}\frac{1}{p!}\frac{v_1^{\frac{n-4+2p}{2}}v_2^{\frac{n-1+2p}{2}}dv_1dv_2}{(2n-2)(2n+2)\ldots(2n+4p-6)}$$

$$= Z_0\sqrt{2\pi}\frac{s_1^n s_2^n}{\kappa^{n+1}}\,2^{n-\frac{5}{2}}\rho z_2 \sum_{p=0}^{p=\infty}\left(\frac{2\rho}{\kappa}\right)^{2p}\frac{1}{p!}\frac{\Gamma\left(\frac{n-2}{2}+p\right)\Gamma\left(\frac{n+1}{2}+p\right)}{(2n-2)(2n+2)\ldots(2n+4p-6)}$$

$$= Z_0\sqrt{2\pi}\frac{s_1^n s_2^n}{\kappa^{n+1}}\,2^{n-\frac{5}{2}}\rho z_2\,\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n+1}{2}\right)F\left(\frac{n-2}{2},\frac{n+1}{2},\frac{n-1}{2},\frac{\rho^2}{\kappa^2}\right)$$

$$= Z_0\sqrt{2\pi}\frac{s_1^n s_2^n}{\kappa^{n+1}}\,2^{n-\frac{5}{2}}\rho z_2\,\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\left(1-\frac{\rho^2}{\kappa^2}\right)^{-\frac{1}{2}n} \times F\left(\tfrac{1}{2},-1,\frac{n-1}{2},\frac{\rho^2}{\kappa^2}\right),$$

by Euler's transformation,

$$= Z_0\sqrt{2\pi}\frac{s_1^n s_2^n}{\kappa^{n+1}}\,2^{n-\frac{5}{2}}\rho z_2\,\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\left(1-\frac{\rho^2}{\kappa^2}\right)^{-\frac{1}{2}n} \times\left(1-\frac{1}{n-1}\frac{\rho^2}{\kappa^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(vii).}$$

It remains now to find n_{z_2}. We have

$$n_{z_2} = \int_0^\infty\int_0^\infty\int_{-\infty}^{+\infty} Z\,d\sigma_1 d\sigma_2 dz_1$$

$$= Z_0\int_0^\infty\int_0^\infty\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{z_1\sigma_1}{s_1}-\rho\frac{z_2\sigma_2}{s_2}\right)^2} dz_1\, e^{-\frac{1}{2}\frac{\sigma_1^2}{s_1^2}} e^{-\frac{1}{2}\frac{\sigma_2^2}{s_2^2}\kappa^2} Q(\sigma_1\sigma_2)^{n-1}\,d\sigma_1 d\sigma_2.$$

Integrating out for dz_1, there results

$$n_{z_2} = Z_0\sqrt{2\pi}\int_0^\infty\int_0^\infty \frac{s_1}{\sigma_1}\,e^{-\frac{1}{2}\frac{\sigma_1^2}{s_1^2}} e^{-\frac{1}{2}\frac{\sigma_2^2}{s_2^2}\kappa^2} Q(\sigma_1\sigma_2)^{n-1}\,d\sigma_1\,d\sigma_2.$$

Changing again to v_1 and v_2 we find

$$n_{z_2} = Z_0\sqrt{2\pi}\,2^{n-\frac{5}{2}}\frac{s_1^n s_2^n}{\kappa^n}\int_0^\infty\int_0^\infty e^{-v_1}e^{-v_2}\sum_0^\infty\left(\frac{2\rho}{\kappa}\right)^{2p}\frac{1}{p!}\frac{v_1^{\frac{n-3}{2}+p}v_2^{\frac{n-2}{2}+p}dv_1dv_2}{(2n-2)(2n+2)\ldots(2n+4p-6)}$$

$$= Z_0\sqrt{2\pi}\,2^{n-\frac{5}{2}}\frac{s_1^n s_2^n}{\kappa^n}\,S\left(\frac{2\rho}{\kappa}\right)^{2p}\frac{1}{p!}\frac{\Gamma\left(\frac{n-1}{2}+p\right)\Gamma\left(\frac{n}{2}+p\right)}{(2n-2)(2n+2)\ldots(2n+4p-6)}$$

$$= Z_0\sqrt{2\pi}\,2^{n-\frac{5}{2}}\frac{s_1^n s_2^n}{\kappa^n}\,\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right) \times\left(1+\frac{\frac{n-1}{2}\frac{n}{2}}{1!(2n-2)}\left(\frac{2\rho}{\kappa}\right)^2+\frac{\frac{n-1}{2}\frac{n+1}{2}\frac{n}{2}\left(\frac{n}{2}+1\right)}{2!(2n-2)(2n+2)}\left(\frac{2\rho}{\kappa}\right)^4+\ldots\right).$$

The series reduces to

$$1+\frac{n}{2}\frac{\rho^2}{\kappa^2}+\frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{1\,.\,2}\frac{\rho^4}{\kappa^4}+\ldots=\left(1-\frac{\rho^2}{\kappa^2}\right)^{-\frac{1}{2}n}.$$

Thus

$$n_{z_2}=Z_0\sqrt{2\pi}\,2^{n-\frac{1}{2}}\frac{s_1^n s_2^n}{\kappa^n}\,\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right)\left(1-\frac{\rho^2}{\kappa^2}\right)^{-\frac{1}{2}n} \quad \ldots\ldots\ldots\text{(viii)}.$$

Dividing (vii) by (viii) we reach, on substituting for κ^2,

$$\tilde{z}_1=\frac{\rho z_2}{\sqrt{1+(1-\rho^2)z_2^2}}\left(\frac{n-1}{n-2}\right)\left(1-\frac{1}{n-1}\frac{\rho^2}{1+(1-\rho^2)z_2^2}\right)\ldots\ldots\ldots\text{(ix)}.$$

This may be put into the simple form

$$\tilde{z}_1=\frac{\rho z_2}{\sqrt{1+(1-\rho^2)z_2^2}}\left(1+\frac{1}{n-2}\frac{(1-\rho^2)(1+z_2^2)}{1+(1-\rho^2)z_2^2}\right) \quad \ldots\ldots\ldots(\text{ix})^{\text{bis}}.$$

This is the regression equation of z_1 on z_2.

We see that if n be finite it is by no means linear. As n grows indefinitely large, it tends to

$$\tilde{z}_1=\frac{\rho z_2}{\sqrt{1+(1-\rho^2)z_2^2}},$$

but, if we remember that the standard deviation of z_2 is $1/\sqrt{n-3}$, z_2^2 will be negligible before z_2, and accordingly we have

$$n\to\infty\,,\quad \tilde{z}_1\to\rho z_2.$$

The form of the curve is algebraically somewhat complicated. It has a point of inflexion at the origin and for asymptotes has the horizontal lines

$$z_1=\pm\frac{n-1}{n-2}\frac{\rho}{\sqrt{1-\rho^2}}.$$

There are further points of inflexion given by

$$z_2^2=\frac{3\rho^2-(n-1)}{(n-1+2\rho^2)(1-\rho^2)},$$

but these will be imaginary if $n-1>3\rho^2$, which it will be if $n=4$, and for practical purposes it is hard to conceive a problem where correlations could be based on samples of 2 or 3. The tangent at the origin, i.e. that at the point of inflexion, is

$$z_1=\frac{n-1-\rho^2}{n-2}\rho z_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(x)}.$$

If $n-1>3\rho^2$ the tangent (x) does not meet again the curve (ix). The general form of (ix) is given diagrammatically in Fig. 1 on p. 416.

Clearly linearity increases more and more as n approaches nearer to infinity. Our figure of the regression line is drawn for the case when n is $=$ or >4, and accordingly the two other points of inflexion vanish.

The accompanying Table II, prepared by Mr E. C. Fieller, indicates how, for various values of ρ, the ordinate z_2 and the size of the sample n, the mean value of $\tilde{z}_1$, differs (i) from the z_1 of the tangent at the point of inflexion, and (ii) from the

line $z_1 = \rho z_2$, the regression straight line. Three values of z_2 are taken respectively equal to once, twice and thrice the standard deviation, $\frac{1}{\sqrt{n-3}}$, of z_2; these will cover the really important part of the regression curve. It will be seen that the deviation from linear regression can be fairly considerable even for a sample of 50.

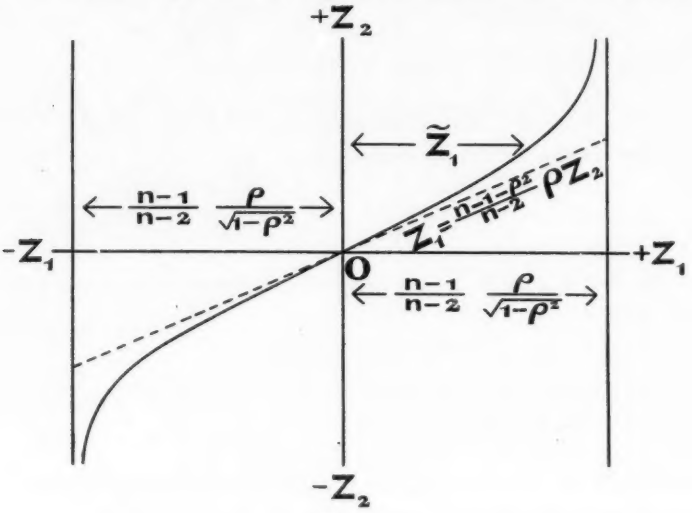


Fig. 1.

(5) *Scedasticity of Arrays.*

We shall now show that the arrays of z_1 for a given value of z_2 are heteroscedastic. In order to obtain the variance of an array we require first to determine

$$n_{z_2} \times {}_{z_2}\mu'_{2,z_1} = \int_{-\infty}^{+\infty}\int_0^\infty\int_0^\infty Z z_1^2 dz_1 d\sigma_1 d\sigma_2$$

$$= Z_0 \int_0^\infty d\sigma_1 \int_0^\infty d\sigma_2 \int_{-\infty}^{+\infty} e^{-\frac{1}{2}u^2}\left(u + \rho z_2 \frac{\sigma_2}{s_2}\right)^2 du \, \frac{s_1^3}{\sigma_1^3} Q$$

$$\times (\sigma_1\sigma_2)^{n-1} e^{-\frac{1}{2}\left(\frac{\sigma_1^2}{s_1^2} + \frac{\sigma_2^2}{s_2^2}\kappa^2\right)},$$

the symbols having the same significance as on p. 413. Hence

$$n_{z_2} \times {}_{z_2}\mu'_{2,z_1} = Z_0 \int_0^\infty d\sigma_1 \int_0^\infty d\sigma_2 \sqrt{2\pi}\left(1 + \rho^2 z_2^2 \frac{\sigma_2^2}{s_2^2}\right)\frac{s_1^3}{\sigma_1^3} \times (\sigma_1\sigma_2)^{n-1}$$

$$\times \underset{p=0}{\overset{p=\infty}{S}} \frac{\rho^{2p}\sigma_1^{2p}\sigma_2^{2p}}{p!(2n-2)(2n+2)\dots(2n+4p-6)\, s_1^{2p} s_2^{2p}} e^{-\frac{1}{2}\left(\frac{\sigma_1^2}{s_1^2} + \frac{\sigma_2^2}{s_2^2}\kappa^2\right)}$$

$$= \frac{\sqrt{2\pi} Z_0 s_1^n s_2^n}{\kappa^n} 2^{n-\frac{1}{2}} \int_0^\infty dv_1 \int_0^\infty dv_2 e^{-v_1 - v_2}\left(1 + \frac{2\rho^2 z_2^2}{\kappa^2} v_2\right)$$

$$\times S\left(\frac{2^2\rho^2}{\kappa^2}\right)^p \frac{1}{p!} v_1^{\frac{n-5}{2}+p} v_2^{\frac{n-2}{2}+p} \frac{1}{(2n-2)(2n+2)\dots(2n+4p-6)}.$$

TABLE II.

Regression Values of $\tilde{z}_1$ on z_2.

	$\rho=\cdot2$			$\rho=\cdot4$			$\rho=\cdot6$			$\rho=\cdot8$		
z_2	$\tilde{z}_1$	$\frac{n-1-\rho^2}{n-2}\rho z_2$	ρz_2	$\tilde{z}_1$	$\frac{n-1-\rho^2}{n-2}\rho z_2$	ρz_2	$\tilde{z}_1$	$\frac{n-1-\rho^2}{n-2}\rho z_2$	ρz_2	$\tilde{z}_1$	$\frac{n-1-\rho^2}{n-2}\rho z_2$	ρz_2
$n=10$												
$1/\sqrt{n-3}$	·079448	·084664	·075593	·158163	·167060	·151186	·235256	·244921	·226779	·309308	·315978	·302372
$2/\sqrt{n-3}$	·136285+	·169328	·151186	·276258	·334121	·302372	·423833	·489842	·453557	·583043	·631957	·604743
$3/\sqrt{n-3}$	·170342	·253992	·226779	·350772	·501181	·453557	·554452	·734763	·680336	·802734	·947935−	·907115−
$n=50$												
$1/\sqrt{n-3}$	·029458	·029756	·029173	·058847	·059367	·058346	·087519	·088686	·087519	·117134	·117567	·116692
$2/\sqrt{n-3}$	·057225−	·059513	·058346	·114730	·118734	·116692	·172796	·177372	·175038	·231704	·235134	·233384
$3/\sqrt{n-3}$	·082056	·089269	·087519	·165377	·178101	·175038	·251319	·266058	·262557	·341430	·352702	·350076
$n=100$												
$1/\sqrt{n-3}$	·020405	·020506	·020307	·040786	·040962	·040614	·061119	·061319	·060921	·081377	·081526	·081228
$2/\sqrt{n-3}$	·040612	·041012	·040614	·080545+	·081924	·081228	·121062	·122637	·121842	·161871	·163052	·162455
$3/\sqrt{n-3}$	·060625	·061518	·060921	·118374	·122886	·121842	·178762	·183956	·182762	·240644	·244578	·243683
$n=500$												
$1/\sqrt{n-3}$	·008980	·008989	·008971	·017958	·017973	·017942	·026931	·026948	·026914	·035898	·035911	·035885−
$2/\sqrt{n-3}$	·017908	·017977	·017942	·035825−	·035945+	·035885−	·053758	·053897	·053827	·071718	·071822	·071770
$3/\sqrt{n-3}$	·026734	·026966	·026914	·053513	·053918	·053827	·080381	·080845−	·080741	·107384	·107733	·107655−

Now $$\frac{2^{2p}}{(2n-2)(2n+2)\dots(2n+4p-6)} = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}+p\right)};$$

hence

$$n_{z_2} \times {}_{z_2}\mu'_{2,z_1} = \frac{\sqrt{2\pi}\, Z_0 {s_1}^n {s_2}^n}{\kappa^n} 2^{n-\frac{1}{2}} \Gamma\left(\frac{n-1}{2}\right)$$

$$\times \sum_0^\infty \left[\frac{\Gamma\left(\frac{n-3}{2}+p\right)}{\Gamma\left(\frac{n-1}{2}+p\right)} \Gamma\left(\frac{n}{2}+p\right) \left(\frac{\rho^2}{\kappa^2}\right)^p \frac{1}{p!} \right.$$

$$\left. + \frac{2\rho^2 {z_2}^2}{\kappa^2} \frac{\Gamma\left(\frac{n-3}{2}+p\right)\Gamma\left(\frac{n}{2}+p+1\right)}{\Gamma\left(\frac{n-1}{2}+p\right)} \left(\frac{\rho^2}{\kappa^2}\right)^p \frac{1}{p!} \right]$$

$$= \sqrt{2\pi}\, Z_0 \frac{{s_1}^n {s_2}^n}{\kappa^n} 2^{n-\frac{1}{2}} \Gamma\left(\frac{n-1}{2}\right)$$

$$\times \sum_0^\infty \frac{1}{\frac{n-3}{2}+p} \Gamma\left(\frac{n}{2}+p\right) \left(\frac{\rho^2}{\kappa^2}\right)^p \frac{1}{p!} \left(1 + \frac{(n+2p)\,\rho^2 {z_2}^2}{\kappa^2}\right) \quad \dots\dots\text{(xi)}.$$

But $$n_{z_2} = \sqrt{2\pi}\, Z_0 \frac{{s_1}^n {s_2}^n}{\kappa^n} 2^{n-\frac{1}{2}} \Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{n}{2}\right) \left(1-\frac{\rho^2}{\kappa^2}\right)^{-n},$$

thus $$ {}_{z_2}\mu'_{2,z_1} = \frac{\left(1-\frac{\rho^2}{\kappa^2}\right)^n}{\Gamma\left(\frac{n}{2}\right)} \sum_0^\infty \frac{\Gamma\left(\frac{n}{2}+p\right)}{(n-3+2p)} \left(\frac{\rho^2}{\kappa^2}\right)^p \frac{1}{p!} \left(1 + \frac{(n+2p)\,\rho^2 {z_2}^2}{\kappa^2}\right) \dots\dots\text{(xii)}.$$

We have two series to consider, namely

$$\sum_0^\infty \frac{\Gamma\left(\frac{n}{2}+p\right)}{\Gamma\left(\frac{n}{2}\right)(n-3+2p)} \left(\frac{\rho^2}{\kappa^2}\right)^p \frac{1}{p!},$$

and

$$\sum_0^\infty \frac{n+2p}{(n-3+2p)} \frac{\Gamma\left(\frac{n}{2}+p\right)}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{\rho^2}{\kappa^2}\right)^p \frac{1}{p!}.$$

The latter

$$= \sum_1^\infty \frac{\Gamma\left(\frac{n}{2}+p\right)}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{\rho^2}{\kappa^2}\right)^p \frac{1}{p!} + \sum_0^\infty \frac{3}{(n-3+2p)} \frac{\Gamma\left(\frac{n}{2}+p\right)}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{\rho^2}{\kappa^2}\right)^p \frac{1}{p!}$$

$$= \left(1-\frac{\rho^2}{\kappa^2}\right)^{-\frac{n}{2}} + 3 \sum_0^\infty \frac{1}{(n-3+2p)} \frac{\Gamma\left(\frac{n}{2}+p\right)}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{\rho^2}{\kappa^2}\right)^p \frac{1}{p!}.$$

Thus we have, if we write the first series Σ,

$$z_3\mu'_{2,z_1} = \frac{\rho^2 z_2^2}{\kappa^2} + \left(1 - \frac{\rho^2}{\kappa^2}\right)^{\frac{n}{2}} \left(1 + \frac{3\rho^2 z_2^2}{\kappa^2}\right)\Sigma \quad \text{.....(xiii).}$$

Now

$$\Sigma = \frac{1}{n-3} + \frac{\frac{1}{2}n}{(n-1)\,1\,!}\frac{\rho^2}{\kappa^2} + \frac{\frac{1}{2}n\left(\frac{1}{2}n+1\right)}{(n+1)\,2\,!}\frac{\rho^4}{\kappa^4} + \dots$$

$$= \int_0^1 u^{n-4}\left(1 - \frac{\rho^2}{\kappa^2}u^2\right)^{-\frac{1}{2}n} du \quad \text{.....(xiv).}$$

The series for Σ converges, since if t_p be the pth term

$$t_{p+1} = \left(t + \frac{n}{2(p+1)}\right)\left(\frac{1}{1+\frac{2}{n+p}}\right)\frac{\rho^2}{\kappa^2}t_p,$$

and the first factor can be made to approach as near to unity as we please by indefinitely increasing p, the second and third factors are always less than unity; thus the series approaches a geometrical series of radix ρ^2/κ^2. It converges, however, far too slowly to be of service for computing. We need to transform the integral into a more rapidly converging series. We put $\frac{\rho^2}{\kappa^2}u^2 = \frac{v}{1+v}$, and find if v_0 be given by $\frac{\rho^2}{\kappa^2} = \frac{v_0}{1+v_0}$ that

$$\Sigma = \int_0^1 u^{n-4}\left(1 - \frac{\rho^2}{\kappa^2}u^2\right)^{-\frac{1}{2}n} du$$

$$= \tfrac{1}{2}\left(\frac{\kappa}{\rho}\right)^{n-3}\int_0^{v_0} v^{\frac{n-5}{2}}(1+v)^{\frac{1}{2}}\,dv.$$

Integrating by parts, raising the power of v, we find

$$\Sigma = \frac{1}{n-3}\left(\frac{\kappa}{\rho}\right)^{n-3}(1+v_0)^{\frac{1}{2}}v_0^{\frac{n-3}{2}}\left(1 - \frac{1}{n-1}\frac{v_0}{1+v_0} - \frac{1}{(n-1)(n+1)}\left(\frac{v_0}{1+v_0}\right)^2\right.$$

$$\left. - \frac{1.3}{(n-1)(n+1)(n+3)}\left(\frac{v_0}{1+v_0}\right)^3 - \frac{1.3.5}{(n-1)(n+1)(n+3)(n+5)}\left(\frac{v_0}{1+v_0}\right)^4 - \text{etc.}\right),$$

a sufficiently converging series for practical purposes.

Substituting for v_0 we have

$$\Sigma = \frac{1}{n-3}\frac{1}{\left(1-\frac{\rho^2}{\kappa^2}\right)^{\frac{n-3}{2}}}\left(1 - \frac{1}{n-1}\frac{\rho^2}{\kappa^2} - \frac{1}{(n-1)(n+1)}\frac{\rho^4}{\kappa^4}\right.$$

$$\left. - \frac{1.3}{(n-1)(n+1)(n+3)}\frac{\rho^6}{\kappa^6} - \frac{1.3.5}{(n-1)(n+1)(n+3)(n+5)}\frac{\rho^8}{\kappa^8} - \text{etc.}\right) \quad \text{.....(xv)}$$

$$= \frac{1}{n-3}\frac{1}{\left(1-\frac{\rho^2}{\kappa^2}\right)^{\frac{n-2}{2}}} \times \Sigma', \text{ say.}$$

Accordingly we have from (xiii)

$$z_3\mu'_{2,z_1} = \frac{\rho^2 z_2^2}{\kappa^2}\left(1 + \frac{3}{n-3}\left(1 - \frac{\rho^2}{\kappa^2}\right)\Sigma'\right) + \frac{1}{n-3}\left(1 - \frac{\rho^2}{\kappa^2}\right)\Sigma' \quad \text{.....(xvi),}$$

TABLE III.

Measurement of the Scedasticity of z_1 for a given z_2. Exact Value and Approximations.

	$\rho = \cdot 2$				$\rho = \cdot 4$			
z_2	True $\sigma_{z_1.z_2}$	Formula (xxi)	$\sqrt{\frac{1-r^2 z_1 z_2}{n-3}}$	$\sqrt{\frac{1-\rho^2}{n-3}}$	True $\sigma_{z_1.z_2}$	Formula (xxi)	$\sqrt{\frac{1-r^2 z_1 z_2}{n-3}}$	$\sqrt{\frac{1-\rho^2}{n-3}}$
$n=10$								
$1/\sqrt{n-3}$	·371586	·371645+	·37132	·370328	·351067	·351273	·35024	·346410
$2/\sqrt{n-3}$	·375630	·375679	,,	,,	·366949	·367142	,,	,,
$3/\sqrt{n-3}$	·379065+	·379106	,,	,,	·381290	·381670	,,	,,
$n=50$								
$1/\sqrt{n-3}$	·142981	·142981	·14298	·142918	·133923	·133923	·13392	·133687
$2/\sqrt{n-3}$	·143318	·143318	,,	,,	·135195−	·135195+	,,	,,
$3/\sqrt{n-3}$	·143802^5	·143802^5	,,	,,	·137055−	·137055+	,,	,,
$n=100$								
$1/\sqrt{n-3}$	·099504	·099504	·09950+	·099483	·093136	·093136	·09314	·093058
$2/\sqrt{n-3}$	·099622	·099622	,,	,,	·093581	·093581	,,	,,
$3/\sqrt{n-3}$	·099805+	·099805+	,,	,,	·094274	·094274	,,	,,
$n=500$								
$1/\sqrt{n-3}$	·043952	·043952	·04395−	·043950−	·041118	·041118	·04112	·041111
$2/\sqrt{n-3}$	·043962	·043962	,,	,,	·041157	·041157	,,	,,
$3/\sqrt{n-3}$	·043979	·043979	,,	,,	·041222	·041222	,,	,,
	$\rho = \cdot 6$				$\rho = \cdot 8$			
$n=10$								
$1/\sqrt{n-3}$	·311304	·311647	·31032	·302372	·238098	·238426	·23839	·226779
$2/\sqrt{n-3}$	·345172	·345559	,,	,,	·288839	·289299	,,	,,
$3/\sqrt{n-3}$	·379422	·379827	,,	,,	·350873	·351495−	,,	,,
$n=50$								
$1/\sqrt{n-3}$	·117150−	·117150+	·11714	·116692	·088122	·088122	·08812	·087519
$2/\sqrt{n-3}$	·119685−	·119685+	,,	,,	·091570	·091571	,,	,,
$3/\sqrt{n-3}$	·123502	·123503	,,	,,	·096991^5	·096992	,,	,,
$n=100$								
$1/\sqrt{n-3}$	·081380	·081380	·08138	·081228	·061123	·061123	·06112	·060921
$2/\sqrt{n-3}$	·082260	·082260	,,	,,	·062307	·062307	,,	,,
$3/\sqrt{n-3}$	·083653	·083653	,,	,,	·064225+	·064225+	,,	,,
$n=500$								
$1/\sqrt{n-3}$	·035897	·035897	·03590	·035885−	·026931	·026931	·02693	·026914
$2/\sqrt{n-3}$	·035976	·035975+	,,	,,	·027035−	·027035−	,,	,,
$3/\sqrt{n-3}$	·036103	·036103	,,	,,	·027206	·027206	,,	,,

and for the variance of the array of z_1's for a given z_2 by (ix)

$$\sigma^2_{z_1.z_2} = \frac{\rho^2 z_2^2}{\kappa^2}\left\{1 + \frac{3}{n-3}\left(1-\frac{\rho^2}{\kappa^2}\right)\Sigma' - \left(\frac{n-1}{n-2}\right)^2\left(1 - \frac{1}{n-1}\frac{\rho^2}{\kappa^2}\right)^2\right\} + \frac{1}{n-3}\left(1-\frac{\rho^2}{\kappa^2}\right)\Sigma' \quad \text{.....(xvii).}$$

It is now possible to compute $\sigma_{z_1.z_2}$ for given values of ρ, z_2 and n, remembering that Σ' is given by

$$\Sigma' = 1 - \frac{1}{n-1}\frac{\rho^2}{\kappa^2} - \frac{1}{(n-1)(n+1)}\frac{\rho^4}{\kappa^4} - \frac{1.3}{(n-1)(n+1)(n+3)}\frac{\rho^6}{\kappa^6} - \frac{1.3.5}{(n-1)(n+1)(n+3)(n+5)}\frac{\rho^8}{\kappa^8} - \text{etc.} \quad \text{.....(xviii),}$$

and that $$\kappa^2 = 1 + (1-\rho^2)z_2^2.$$

It is clear, however, that $\sigma^2_{z_1.z_2}$ is a fairly involved function of z_2, or, in other words, the system is far from homoscedastic. Table III gives the true values of $\sigma_{z_1.z_2}$ for selected values of ρ, z_2 and n in the first column of each section corresponding to a given ρ; in the third column is given the homoscedastic value, i.e.

$$\sigma'_{z_1.z_2} = \frac{1}{\sqrt{n-3}}\sqrt{1-r^2_{z_1z_2}},$$

where $r_{z_1z_2}$ is the correlation of z_1 and z_2, and in the fourth column the homoscedastic value

$$\sigma''_{z_1.z_2} = \frac{1}{\sqrt{n-3}}\sqrt{1-\rho^2},$$

where ρ is the correlation in the parent population. Finally in the second column is given the approximation to $\sigma_{z_1.z_2}$ now to be found, where we assume that terms of the order $\frac{1}{(n-3)^4}$ may be neglected.

To find an Approximation for $\sigma^2_{z_1.z_2}$ in terms of $1/(n-3)$.

We first find $\tilde{z}_1^2$,

$$\tilde{z}_1^2 = \left(\frac{\rho}{\kappa}\right)^2 z_2^2\left(\frac{n-1}{n-2}\right)^2\left(1 - \frac{1}{n-1}\left(\frac{\rho}{\kappa}\right)^2\right)^2.$$

Now

$$\left(\frac{n-1}{n-2}\right)^2 = \left(\frac{1+\frac{2}{n-3}}{1+\frac{1}{n-3}}\right)^2 = 1 + \frac{2}{n-3} - \frac{1}{(n-3)^2} + \frac{0}{(n-3)^3},$$

and

$$\left(1 - \frac{1}{n-1}\frac{\rho^2}{\kappa^2}\right)^2 = 1 - \frac{2}{n-1}\frac{\rho^2}{\kappa^2} + \frac{1}{(n-1)^2}\frac{\rho^4}{\kappa^4}$$

$$= 1 - \frac{2}{n-3}\frac{\rho^2}{\kappa^2} + \frac{1}{(n-3)^2}\frac{\rho^2}{\kappa^2}\left(4 + \frac{\rho^2}{\kappa^2}\right) - \frac{4}{(n-3)^3}\frac{\rho^2}{\kappa^2}\left(2 + \frac{\rho^2}{\kappa^2}\right).$$

Combining these we have

$$\bar{z}_1^2 = \frac{\rho^2 z_2^2}{\kappa^2}\left(1 + \frac{2}{n-3}\left(1 - \frac{\rho^2}{\kappa^2}\right) - \frac{1}{(n-3)^2}\left(1 - \frac{\rho^4}{\kappa^4}\right) + \frac{2}{(n-3)^3}\frac{\rho^2}{\kappa^2}\left(1 - \frac{\rho^2}{\kappa^2}\right)\right) \quad \text{(xix)*}.$$

In the same manner we find

$$\frac{1}{n-3}\left(1 - \frac{\rho^2}{\kappa^2}\right)\Sigma' = \frac{1}{n-3}\left(1 - \frac{\rho^2}{\kappa^2}\right)\left(1 - \frac{1}{n-3}\frac{\rho^2}{\kappa^2} + \frac{1}{(n-3)^2}\frac{\rho^2}{\kappa^2}\left(2 - \frac{\rho^2}{\kappa^2}\right)\right) \quad \text{(xx)}.$$

Substituting (xix) and (xx) in (xvii) we reach, after some reductions,

$$\sigma^2_{z_1 . z_2} = \frac{1}{n-3}\frac{(1-\rho^2)(1+z_2^2)^2}{(1+(1-\rho^2)z_2^2)^2} \times \left\{1 - \frac{1}{n-3}\rho^2\frac{1 - z_2^2(1-\rho^2)}{1+(1-\rho^2)z_2^2} + \frac{1}{(n-3)^2}\frac{\rho^2(\rho^2 + 2(1-\rho^2)(1+(1+\rho^2)z_2^2))}{(1+(1-\rho^2)z_2^2)^2}\right\} \quad \text{.....(xxi)}.$$

This is the expression of $\sigma^2_{z_1 . z_2}$ up to the third order terms in $\frac{1}{n-3}$.

It will be remarked on examination of Mr Fieller's table, Table III, that after $n = 50$, the approximate formula (xxi) agrees with the true value of $\sigma_{z_1 . z_2}$ practically to a unit in the sixth decimal place. For statistical practice it is really efficient down to $n = 25$, i.e. it will only differ in the fifth decimal place. For lower values of n it will be needful to evaluate the full series Σ' of formula (xviii). We note further that the distribution is not adequately homoscedastic even for $n = 500$, the distribution of z_1 for a given z_2 continues to increase in variability as we increase z_2. Of the two suggested formulae for homoscedastic values that for which we use the correlation of z_1 and z_2 gives a better result than that for which we use the correlation of the parent population.

Clearly the non-linearity of the regression and the heteroscedasticity of the arrays are not in favour of using z_1 and z_2 as variates in samples drawn from a parent population with correlated variables. The investigation of the probability that an observed z_1 should be associated with an observed z_2, if the parental population had known means and a given correlation, would require much arithmetical labour.

I have to thank heartily my colleague Mr E. C. Fieller for his help in the preparation of this paper.

* This Equation for samples of 25 and over will give very accurately the mean value of $\bar{z}_1$ for a given z_2.

MISCELLANEA.

I. Note on Tests for Normality.

In an earlier part of the present volume of *Biometrika* I have given certain approximate tables of the 5 °/₀ and 1 °/₀ points for $\sqrt{\beta_1}$ and β_2 in sampling from a normal population. The results were based on expansions in series of inverse powers of n, the sample size, which it had been possible to derive as far as the terms in n^{-3}*. Since the publication of this paper Dr R. A. Fisher has been able to obtain exact expressions for the moment-coefficients of the sampling distribution of these two constants in the case of a normal population†. The quantities with which he deals are

$$\gamma = k_3 k_2^{-\frac{3}{2}} = \frac{\sqrt{n(n-1)}}{n-2}\sqrt{\beta_1} \quad \text{and} \quad \delta = k_4 k_2^{-2} = \frac{n^2-1}{(n-2)(n-3)}\left\{\beta_2 - \frac{3(n-1)}{n+1}\right\} \qquad \text{.....(1),}$$

and from these relations, and making use of the value of $\kappa(4^4)$ given above by Wishart‡, it is possible to obtain the following results:

Distribution of $\sqrt{\beta_1}$.

$$\sigma^2_{\sqrt{\beta_1}} = \frac{6(n-2)}{(n+1)(n+3)} \qquad \text{.....(2),}$$

$$B_2(\sqrt{\beta_1}) = 3 + \frac{36(n-7)(n^2+2n-5)}{(n-2)(n+5)(n+7)(n+9)} \qquad \text{.....(3),}$$

$$B_4(\sqrt{\beta_1}) = 15 + \frac{540\{n^7+60n^6-131n^5-2798n^4-3629n^3+21352n^2+32943n-70070\}}{(n-2)^2(n+5)(n+7)(n+9)(n+11)(n+13)(n+15)} \qquad \text{...(4).}$$

Distribution of β_2.

$$\text{Mean } \beta_2 = \frac{3(n-1)}{n+1} \qquad \text{.....(5),}$$

$$\sigma^2_{\beta_2} = \frac{24n(n-2)(n-3)}{(n+1)^2(n+3)(n+5)} \qquad \text{.....(6),}$$

$$B_1(\beta_2) = \frac{216}{n}\frac{(n+3)(n+5)(n^2-5n+2)^2}{(n-3)(n-2)(n+7)^2(n+9)^2} \qquad \text{.....(7),}$$

$$B_2(\beta_2) = 3 + \frac{36(15n^6-36n^5-628n^4+982n^3+5777n^2-6402n+900)}{n(n-3)(n-2)(n+7)(n+9)(n+11)(n+13)} \qquad \text{.....(8).}$$

As a check on the previous work it is satisfactory to find that the expressions (2), (3), (6), (7) and (8) give on expansion as far as the terms in n^{-3} exactly the same coefficients as those contained in equations (10), (11), (21), (22) and (23) of my earlier paper. For the standard errors of $\sqrt{\beta_1}$ and β_2 the approximate expressions that I had used at $n=50$ and $n=100$ respectively are identical with the true values to four places of decimals. Further the values of $B_2(\sqrt{\beta_1})$ are as follows:

	$n=50$	$n=75$	$n=100$
True B_2	3·452	3·351	3·284
Value used in making tables	3·46	3·35	3·28

* The results were based on the investigations of R. A. Fisher, *Proc. Lond. Math. Soc.* (2), Vol. xxx. (1929), pp. 199—238, and J. Wishart, *Biometrika*, Vol. xxii. pp. 224—238.

† *Proceedings of the Royal Society*, Series A, Vol. 130, No. A 812, pp. 16—28.

‡ P. 233, Equation (15).

And for the distribution of β_2:

	$n=100$	$n=150$		$n=100$	$n=150$
True B_1	1·631	1·192	True B_2	6·774	5·826
Value used	1·63	1·19	Value used	6·85	5·84

Although I have not refitted the curve, I have little doubt that the error in B_2 (β_2) for $n=100$ could only affect the values tabled for the 5 °/₀ and 1 °/₀ points at the most by two units in the second decimal place, and probably only by one unit. Any approximation which may be found to exist will therefore not be due to the use of the first four terms in the series instead of the true values of the moment-coefficients, but to the assumption that Type VII and Type IV curves may be used to find the two probability limits. This point cannot be completely solved until the actual frequency laws for $\sqrt{\beta_1}$ and β_2 have been found.

EGON S. PEARSON.

II. Some recent Researches in the Theory of Statistics and Actuarial Science. By J. F. STEFFENSEN. Cambridge: published for the Institute of Actuaries, at the University Press, 1930. Price 5*s.* net.

THIS little volume gives, in a somewhat extended form, the substance of the three lectures delivered by Professor Steffensen for the University of London in the spring of 1930, and will enable them to reach, as they deserve, a much wider audience. Its modest bulk of fifty-two pages is packed with matter.

Professor Steffensen took as the general subject of his lectures some of the efforts he had made "to introduce more rigour into certain questions of theoretical statistics and actuarial science." In mathematics we have a science which investigates the relations between numbers. Observations may contradict each other, but mathematical relations are not allowed to contain contradictions: theory must be presented in such a form that the theoretical relations or assumptions contain no contradictions. The first lecture is devoted to showing how we may be led astray by neglect of this principle. The opening sections make a critical examination of the notion of *Biometric Functions* (Life Table functions). A number of interesting inequalities are obtained, and some common but loose modes of statement or assumption, e.g. the assumption of an "oldest possible" age at which the l_x column abruptly terminates, come in for useful discussion. The author then turns to the thorny question of "presumptive values" of frequency constants, taking as an illustration presumptive values of the moment-coefficients. Here we are brought up rather sharply by an apparent paradox. The mathematical expectation of the second moment-coefficient *about the mean* in a sample of n is $(n-1)/n$ times the second moment-coefficient in the universe sampled: hence the not-infrequently used formula $n/(n-1)$ times the sample value for the "presumptive value" in the universe. But the mathematical expectation of a moment-coefficient of any order *about a fixed origin* is identical with the value in the universe; and the value about the mean is expressible in terms of the values about a fixed origin. The presumptive values are therefore identical with those in the sample. Professor Steffensen concludes (p. 20): "It appears thus that neither of the two systems of presumptive values of frequency-constants...is free from contradictions, and that a strong case can be made even against the time-honoured Gaussian formula $\overline{m}_2 = \frac{n}{n-1} m_2$. If, on the other hand, we use the uncorrected [sample moments] as the best available approximations to [the moments in the universe] we are at least sure that no contradictions can ever be met with." The conclusion is comforting to one who has always worked with the sample values. But in this case it looks as if precisely the same assumptions led to contradictory conclusions, and Professor Steffensen does not seem to show how they do so. What is the source of the discrepancy?

In the second lecture the author considers problems of approximation and interpolation. Two objects of interpolation are distinguished. When the problem is to find the value of a *defined* function for a certain value of the argument, the function being tabulated for certain other arguments, the problem is one of *approximation*. When the problem is to fill up in a reasonable way a gap in a series of values of an *undefined* function, the process is more analogous to graduation and, it is suggested, might be termed *intercalation*. We are here on different ground, and it is a question of "plausibility" rather than "accuracy" of the result. Illustrations are drawn from the life-table and again some useful inequalities are obtained.

Mathematics are also used for describing facts of observation. Formulae used for the purpose may be empirical, or may be based on theoretical considerations, and theoretical reasons may be given for believing that one formula is likely to fit the facts better than another. Theoretical foundations are therefore of importance, and frequency-functions are chosen as the subject of the third lecture. Professor Karl Pearson's methods and those associated mainly with the names of Thiele, Charlier and Bruns are discussed and contrasted. One of the earliest pupils of Professor Pearson may perhaps be allowed the pleasure of citing from the fifth paragraph of the lecture (p. 35): "It is the lasting merit of Professor Karl Pearson of the University of London to have pointed out convincingly that the *natural source* of practically useful types of frequency-functions is the elementary calculus of probabilities. What nature does in producing a new individual, practically comes to the same thing as drawing from various urns and mixing up the results. This explains the great success of Pearson's types." And again from the conclusion (p. 48): "We are therefore inclined to think that the apparent generality of (28) [the general series] is rather a disadvantage than otherwise, and that Pearson's types are as a rule preferable."

Every statistician who is interested in theory should possess the volume.

G. Udny Yule.

III. A Problem in Probability.

Positive quantities $x_1, x_2, \ldots x_n$ are taken at random subject to the conditions

I $$a_1x_1 \geqslant a_2x_2 \geqslant a_3x_3 \geqslant \ldots \geqslant a_nx_n \qquad (a_i>0),$$

II $$a_1\mathfrak{a}_1x_1 + a_2\mathfrak{a}_2x_2 + a_3\mathfrak{a}_3x_3 + \ldots + a_n\mathfrak{a}_nx_n = 1 \qquad (\mathfrak{a}_i>0).$$

What are the mean values of $x_1, x_2, \ldots x_n$?

Represent the set of quantities $x_1, x_2, \ldots x_n$ by the point $P(x_1, x_2, \ldots x_n)$ in S_n. All points corresponding to sets satisfying the given conditions lie in Γ, the $(n-1)$-dimensional simplex cut out by the primes

(1) $$\begin{cases} a_1x_1 = a_2x_2 \\ a_2x_2 = a_3x_3 \\ \ldots\ldots\ldots\ldots \\ a_{n-1}x_{n-1} = a_nx_n \\ a_nx_n = 0, \end{cases}$$

from the prime

(2) $$a_1\mathfrak{a}_1x_1 + a_2\mathfrak{a}_2x_2 + \ldots + a_n\mathfrak{a}_nx_n = 1.$$

If we omit the rth of equations (1) and solve the remainder with (2), we get for the coordinates of P_r, the rth vertex of Γ,

(3) $$x_i = 1 \Big/ a_i \sum_{p=1}^{r} \mathfrak{a}_p \qquad (i=1 \ldots r),$$

$$= 0 \qquad (i>r).$$

Subject to conditions involved in I, $x_1, x_2, \ldots x_{n-1}$ may be taken at random, and x_n is then determined by II. The chance that x_1 should lie between x_1 and x_1+dx_1, x_2 between x_2 and

$x_2+dx_2, \ldots x_{n-1}$ between x_{n-1} and $x_{n-1}+dx_{n-1}$ is proportional to $dx_1 dx_2 \ldots dx_{n-1}$, which is the content of the hyper-rectangle in which the projection of P on $x_n=0$ must lie. But the content of this hyper-rectangle is proportional to the content of the portion of Γ of which it is the orthogonal projection.

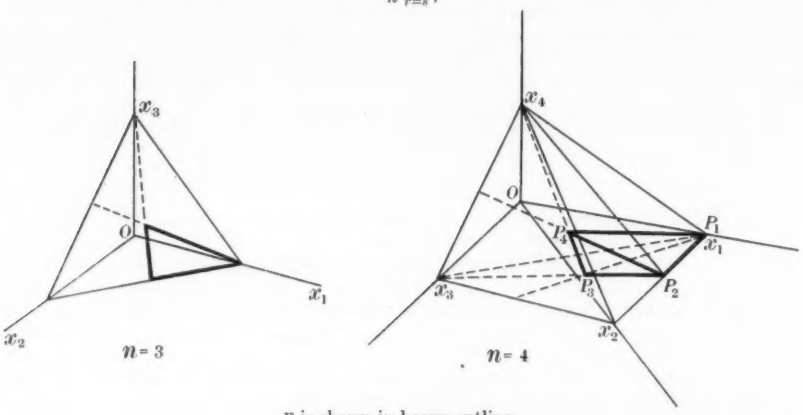
It follows that all positions of P in Γ are equally likely, so that the mean values $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_n$ of $x_1, x_2, \ldots x_n$ are the coordinates of the centroid of Γ; thus

$$\bar{x}_s = \frac{1}{n} \sum_{r=s}^{n} \left(1 \Big/ a_s \sum_{p=1}^{r} a_p \right).$$

Corollary. If we put $a_1 = a_2 = \ldots = a_n = 1 = \alpha_1 = \alpha_2 = \ldots = \alpha_n$, we get, for the mean values of positive quantities $x_1, x_2, \ldots x_n$ chosen subject to the conditions

$$\begin{cases} x_1 \geqq x_2 \geqq x_3 \ldots \geqq x_n \\ x_1 + x_2 + x_3 + \ldots + x_n = 1, \end{cases}$$

$$\bar{x}_s = \frac{1}{n} \sum_{r=s}^{n} \frac{1}{r}.$$



Γ is shown in heavy outline.

This is the result that Laplace uses for his theory of voting, when he says*: "Donnons à chaque votant, une urne qui renferme un nombre infini de boules; et supposons qu'il les distribue sur les diverses propositions, en raison des probabilités respectives qu'il leur attribue.... Le problème se réduit donc à déterminer les combinaisons dans lesquelles les boules seront réparties, de manière qu'il y en ait plus sur la première proposition du billet, que sur la seconde; plus sur la seconde que sur la troisième, etc.; à faire les sommes de tous les nombres de boules, relatifs à chaque proposition dans ces diverses combinaisons; et à diviser cette somme, par le nombre des combinaisons: les quotiens seront les nombres de boules, que l'on doit attribuer aux propositions sur un billet quelconque. On trouve par l'analyse, qu'en partant de la dernière proposition, pour remonter à la première; ces quotiens sont entre eux, comme les quantités suivantes: 1° l'unité divisée par le nombre des propositions; 2° la quantité précédente augmentée de l'unité divisée par le nombre des propositions moins une; 3° cette seconde quantité augmentée de l'unité divisée par le nombre des propositions moins deux; et ainsi du reste."

It is worth noticing that Laplace's wording, as it stands, is somewhat ambiguous; moreover, as Professor Pearson points out, the assumption of equal probability for all modes of division is hardly likely to be justified in practice.

* *Essai philosophique sur les probabilités.*

Laplace insists on his voters' dividing up all their balls between the various proposals, but the result holds without this condition. An argument similar to that used above shows that the mean values $\bar{x}_s$ of quantities x_s chosen at random subject to the conditions

I $$a_1x_1 \geqslant a_2x_2 \geqslant a_3x_3 \geqslant \ldots \geqslant a_nx_n \qquad (a_i>0),$$

and II′ $$a_1\alpha_1x_1 + a_2\alpha_2x_2 + a_3\alpha_3x_3 + \ldots + a_n\alpha_nx_n \leqslant 1 \qquad (\alpha_i>0),$$

are the coordinates of the centroid of the n-dimensional simplex $OP_1P_2 \ldots P_r \ldots P_n$. Thus

$$\bar{x}_s = \frac{1}{n+1} \sum_{r=s}^{n} \left(1 \Big/ a_s \sum_{p=1}^{r} \alpha_p\right),$$

so that the generalization of condition II does not alter the ratios of the mean values.

E. C. Fieller.

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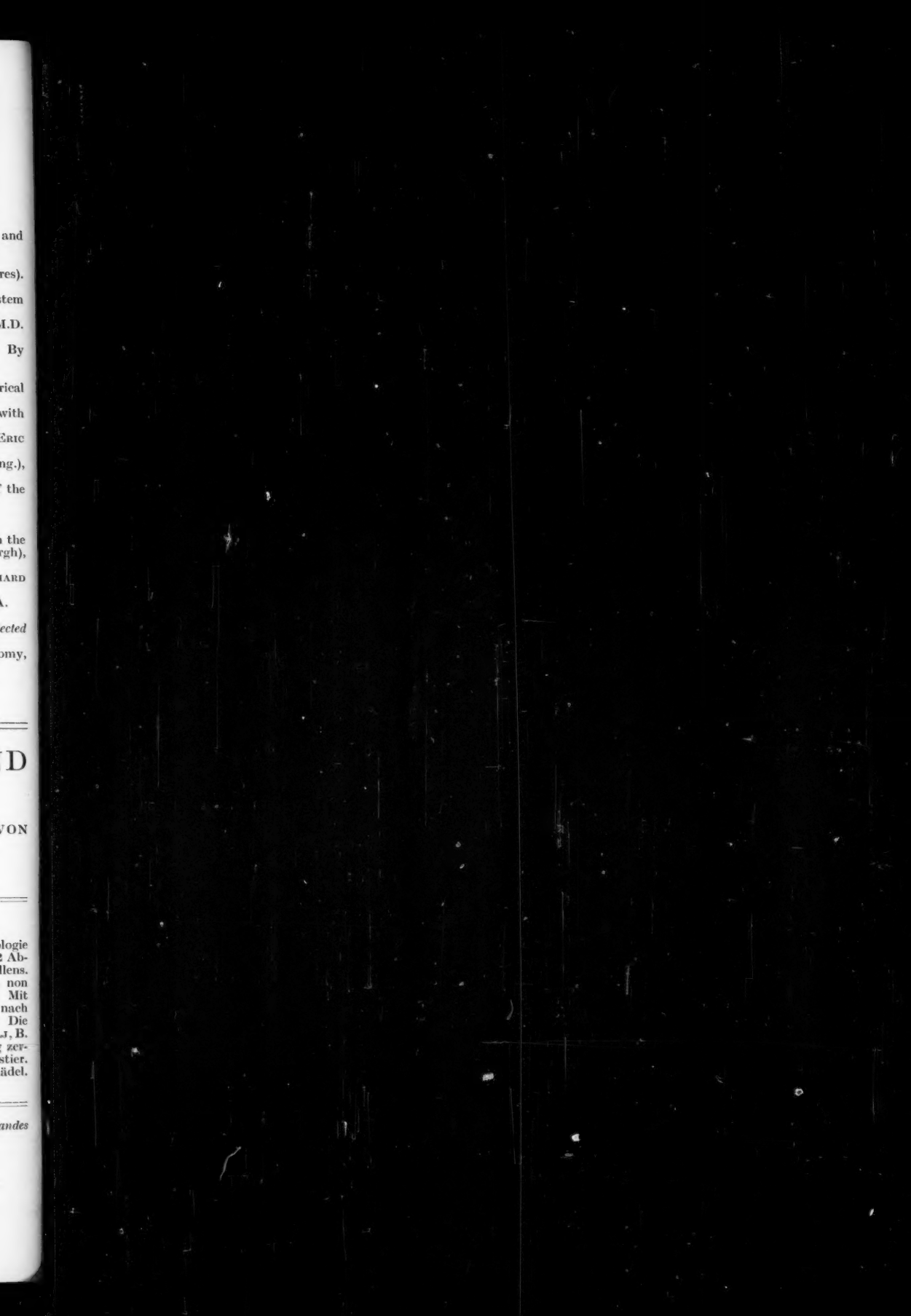

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